A Lecture on Associative Varieties

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Abstract

We state results from noncommutative deformation theory of modules over an associative k-algebra A necessary for this work. We define a set of A-modules containing the simple modules whose elements we call spectral, $\operatorname{aSpec} A$, for which there exists a topology where the simple modules are the closed points. Applying results from deformation theory we prove that there exists a sheaf of associative rings $\mathcal{O}_X$ on the topological space $X = \operatorname{aSpec} A$ giving it the structure of a locally ringed space. In general, an associative variety X is a ringed space with an open covering $\{U_i = \operatorname{aSpec} A_i\}_{i \in I}$. When A is a commutative k-algebra, $\operatorname{aSpec} A \simeq \operatorname{Spec} A$, and so the category $\mathbf{aVar}_k$ of associative varieties is an extension of the category of varieties $\mathbf{Var}_k$, i.e. there exists a faithfully flat functor $I : \mathbf{Var}_k \to \mathbf{aVar}_k$. Our main result says that any associative variety X is $\operatorname{aSpec}(\mathcal{O}_X(X))$ for the k-algebra $\mathcal{O}_X(X)$, and so any study of varieties can be reduced to the study of the associative algebra $\mathcal{O}_X(X)$. (We make Geometry Algebraic again.)