General relativistic bubble growth in cosmological phase transitions

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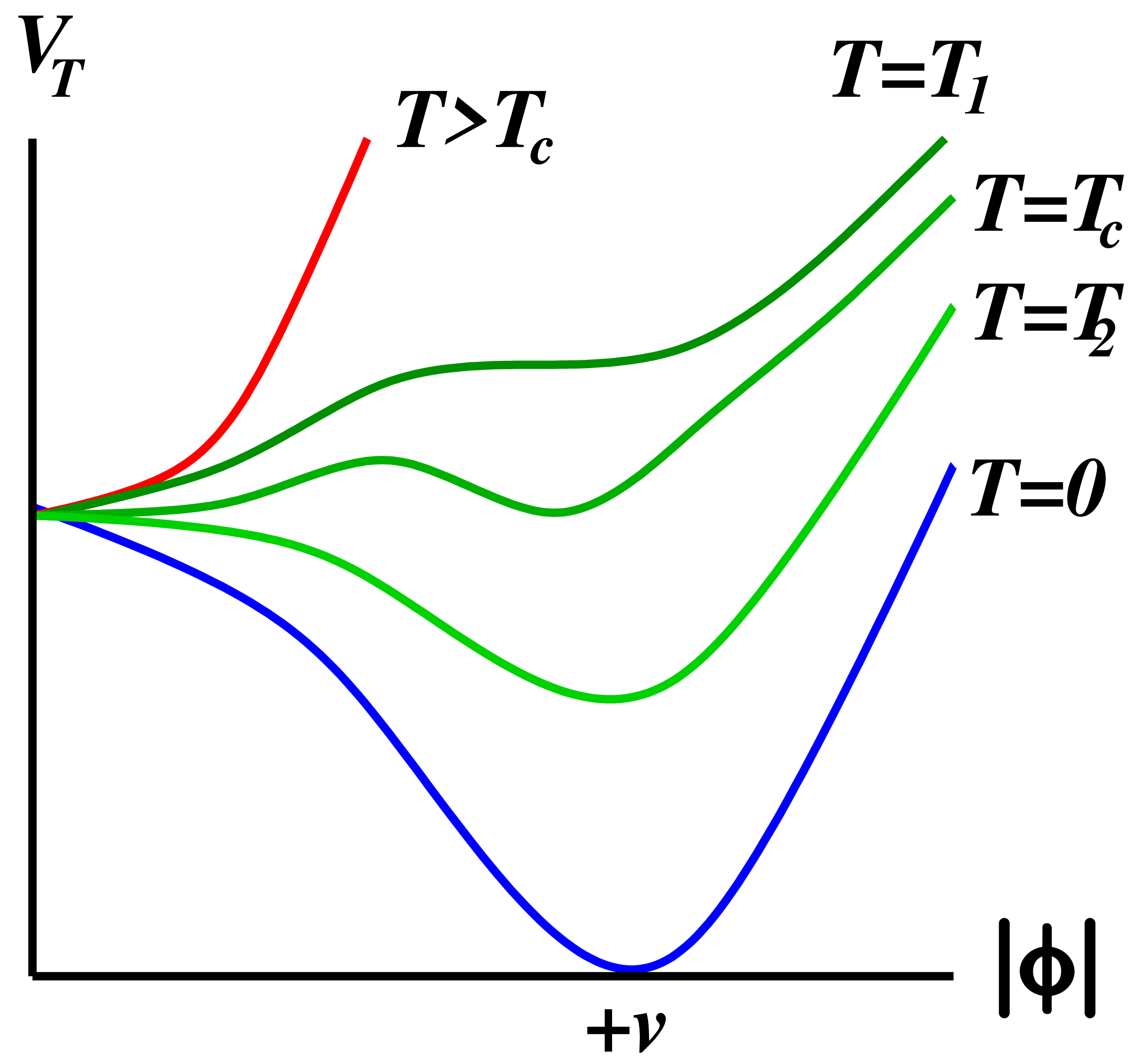
First order phase transitions (FOPT) in the early Universe

M.Hindmarsh et al. (2020), arXiv:2008.09136v2

- Phase transitions are a generic feature of many gauge field theories

- Usually described by a scalar field ϕ with free energy $\mathscr{F}(\phi, T)$



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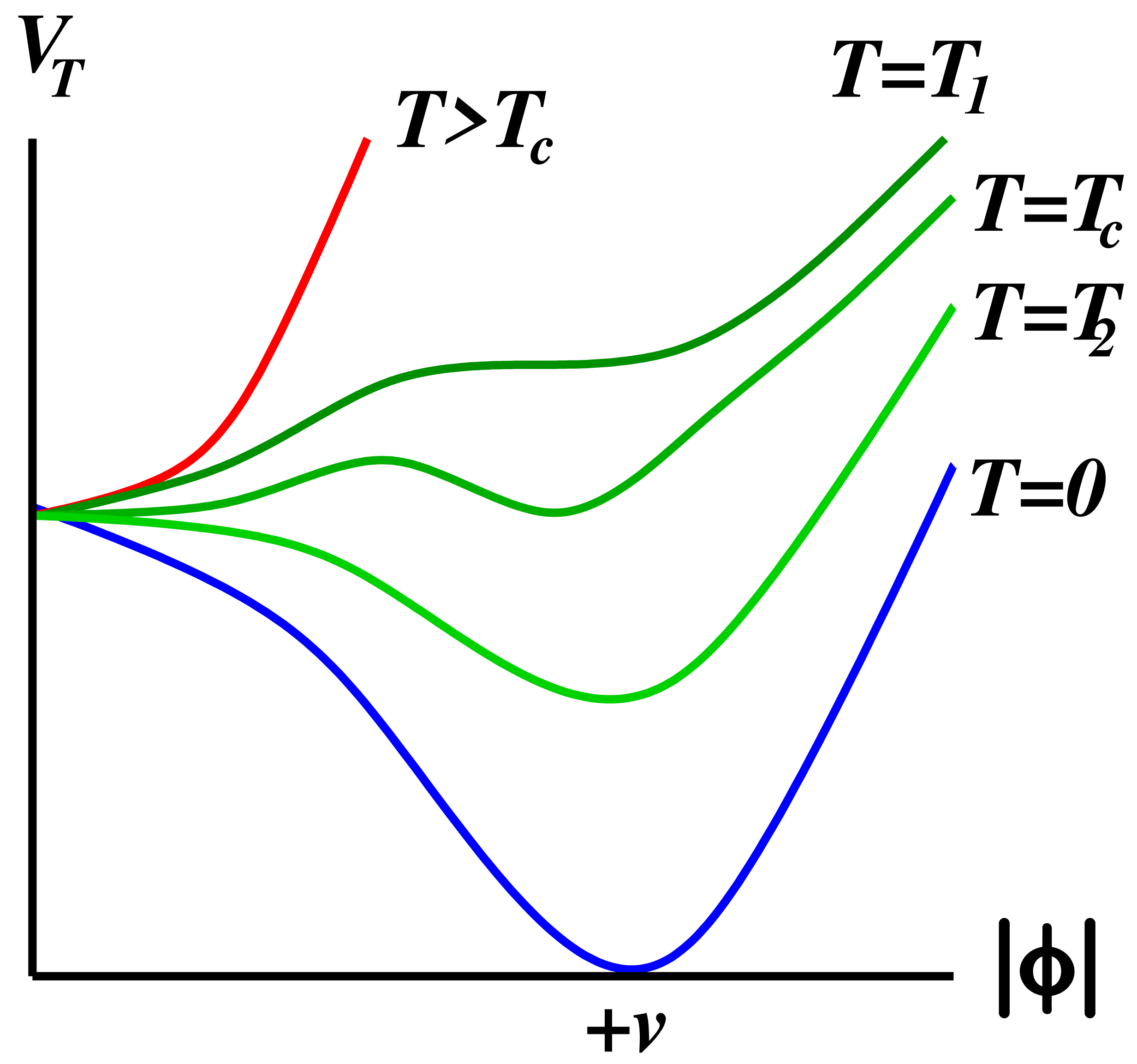

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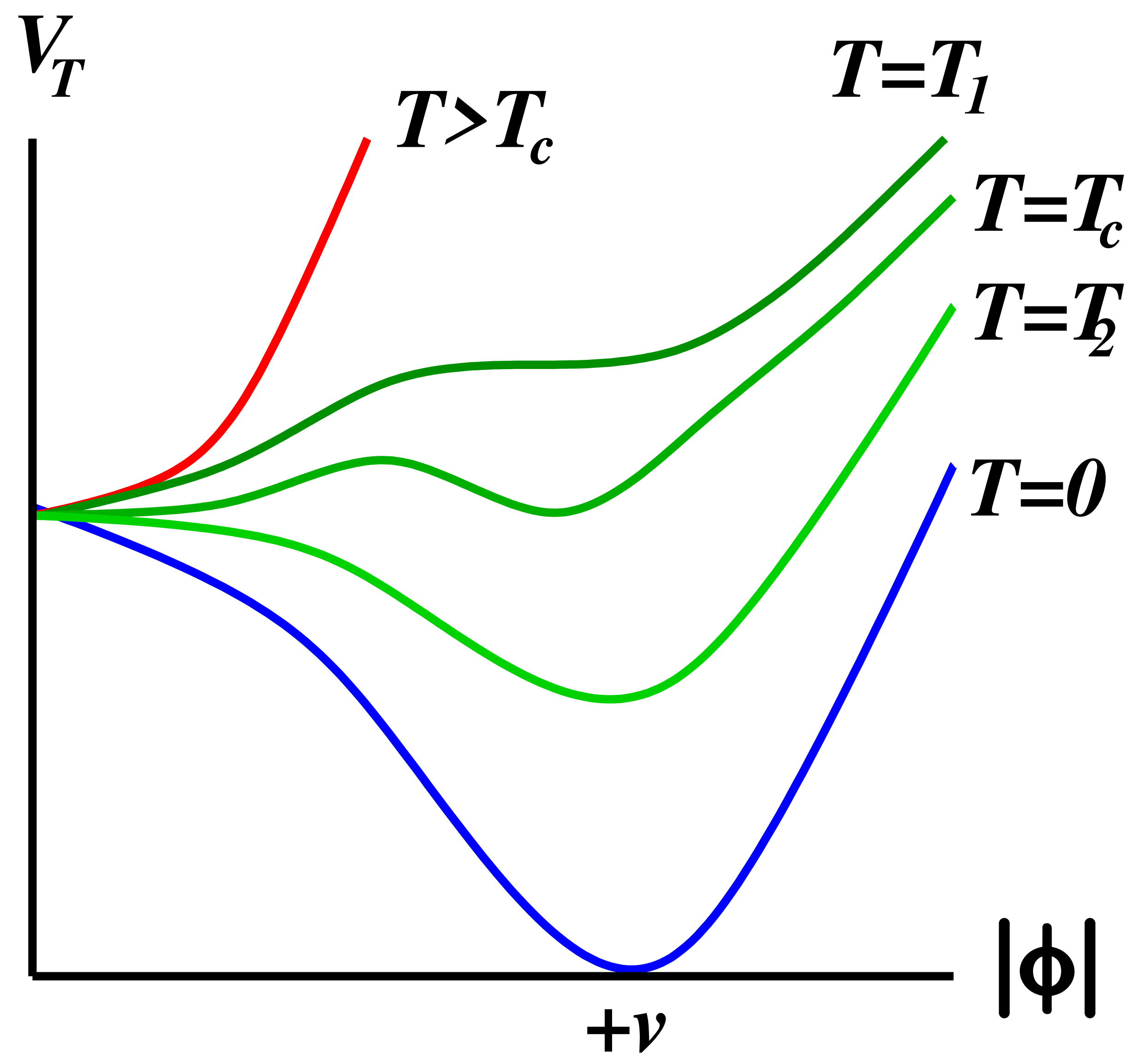
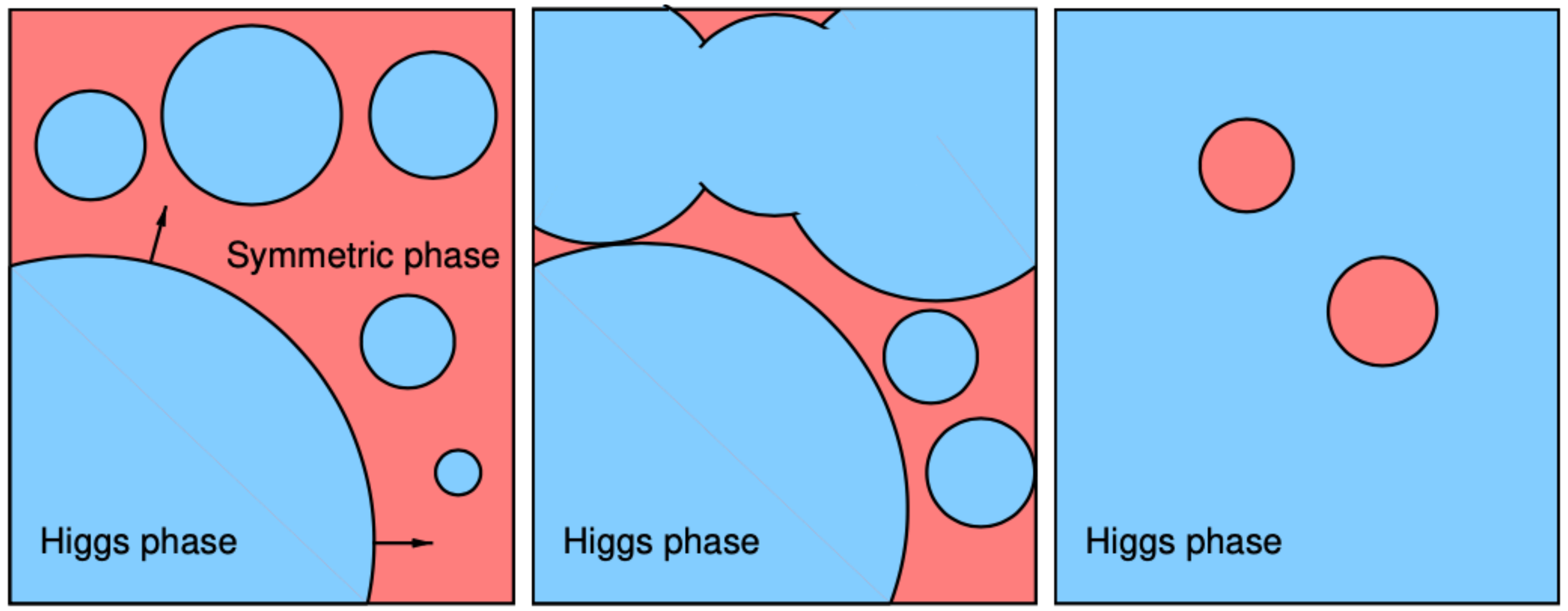
- Phase transitions are a generic feature of many gauge field theories
- Usually described by a scalar field ϕ with free energy $\mathscr{F}(\phi, T)$
- FOPT: just below T_c the field ϕ is in a metastable phase
- Thermal and quantum fluctuations allow the nucleation of bubbles of the stable phase
- Bubbles expand and merge until filling up the entire Universe





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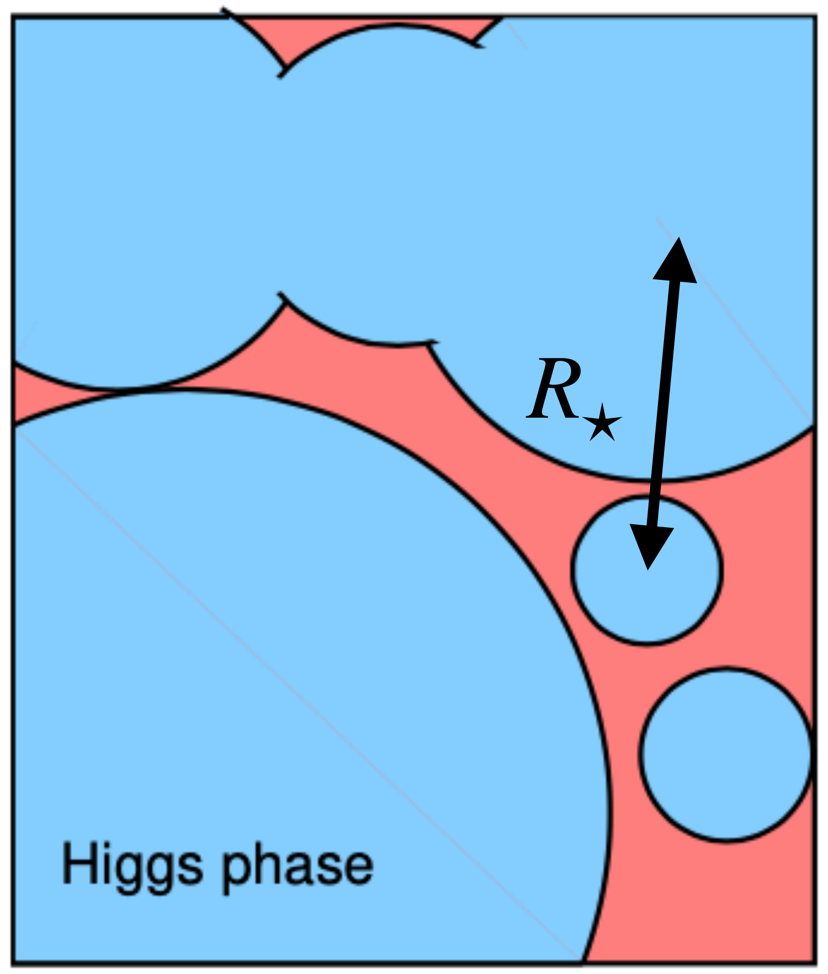
Goal

Hydrodynamic description of a single expanding bubble:

- So far: expansion on a flat Minkowski spacetime ($R_\star \ll H_\star^{-1}$)
- In slow FOPT the timescale of the expansion is of the order of Hubble time ($R_\star \sim H_\star^{-1}$)

→ Need for the full general relativistic treatment



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$R_\star$: mean bubble spacing after nucleation of all bubbles

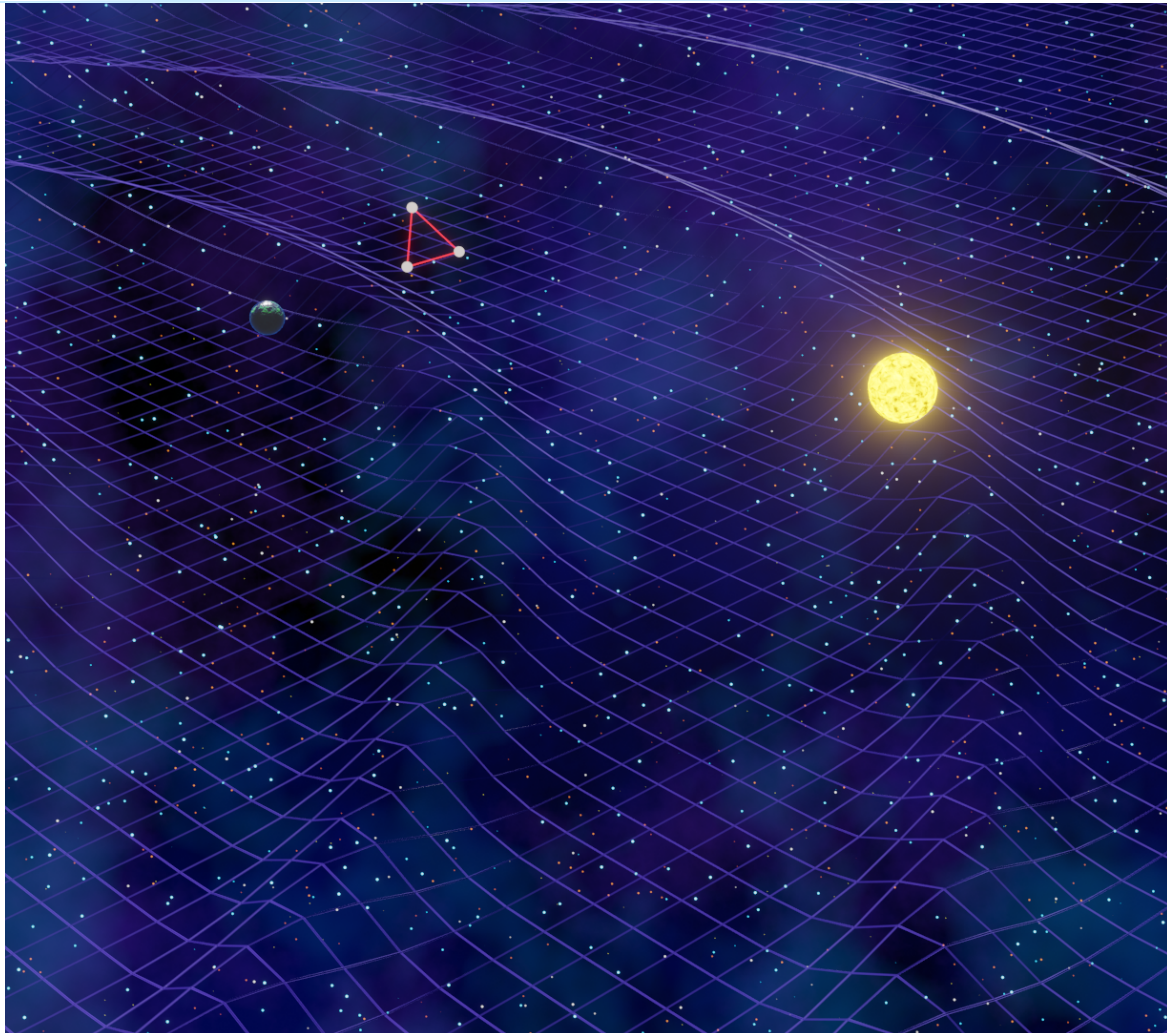
$H_\star^{-1}$: Hubble radius at the time when $1/e$ of metastable phase remains

Motivations

FOPT are a source of the stochastic background of gravitational waves

FOPT at the EW scale (~ 100 GeV) are experimentally interesting for the LISA mission

~ 0.1 mHz - 10 Hz



Credit: Anna Kormu

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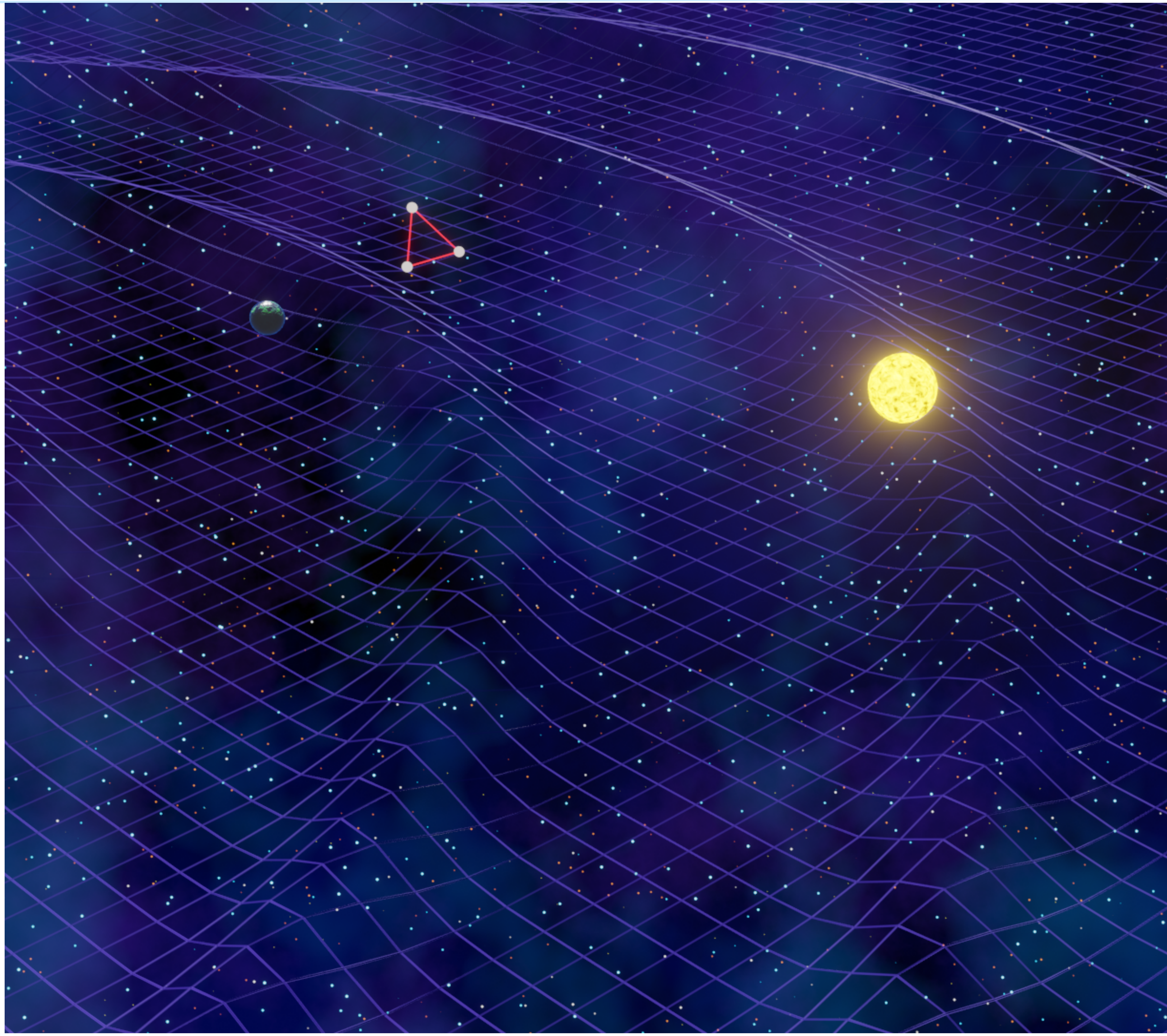
- Energy density in gravitational waves sourced by sound waves $\Omega_{sw} \propto (R_\star H_\star)^n$ $\quad 1 \leq n \leq 2$

C. Caprini et al. (2016),arXiv:1512.06239v2



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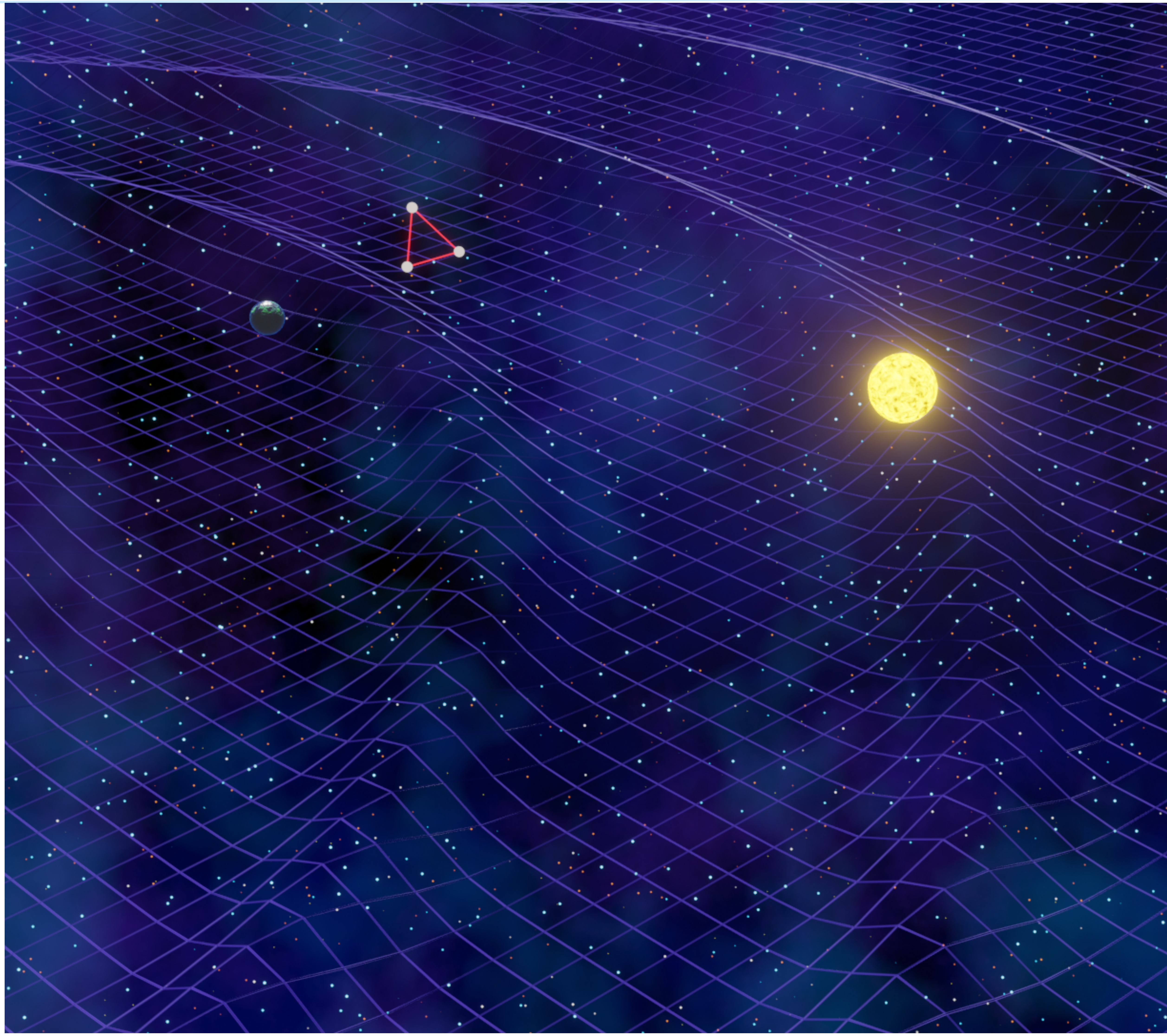
- Cosmological scalar perturbations Φ induce secondary gravitational waves that become important in the limit of large bubbles

$$\frac{\partial_i \Phi \partial_j \Phi}{T_{ij}^{TT}} \sim (HR)^2 \left(\frac{\delta e}{e} \right)$$



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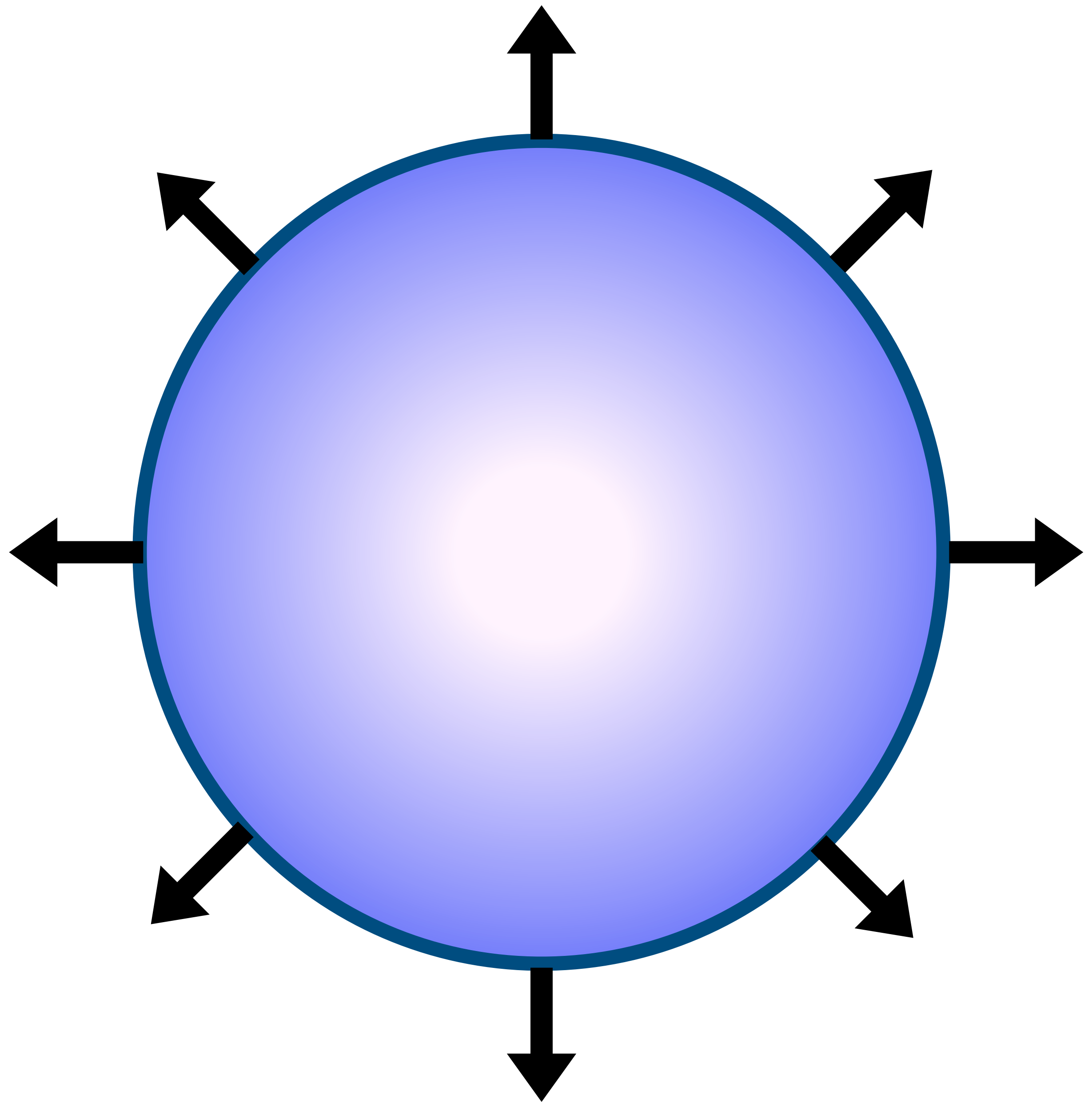
Hydrodynamic description of a single expanding bubble

- Spherical symmetry: $ds^2 = -a^2 dt^2 + b^2 dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$ Misner & Sharp (1964) Phys.Rev 136 B571

$$T^{\mu\nu} = w u^\mu u^\nu + p g^{\mu\nu}, \qquad u^\mu = \frac{1}{a}\delta^{\mu 0}$$



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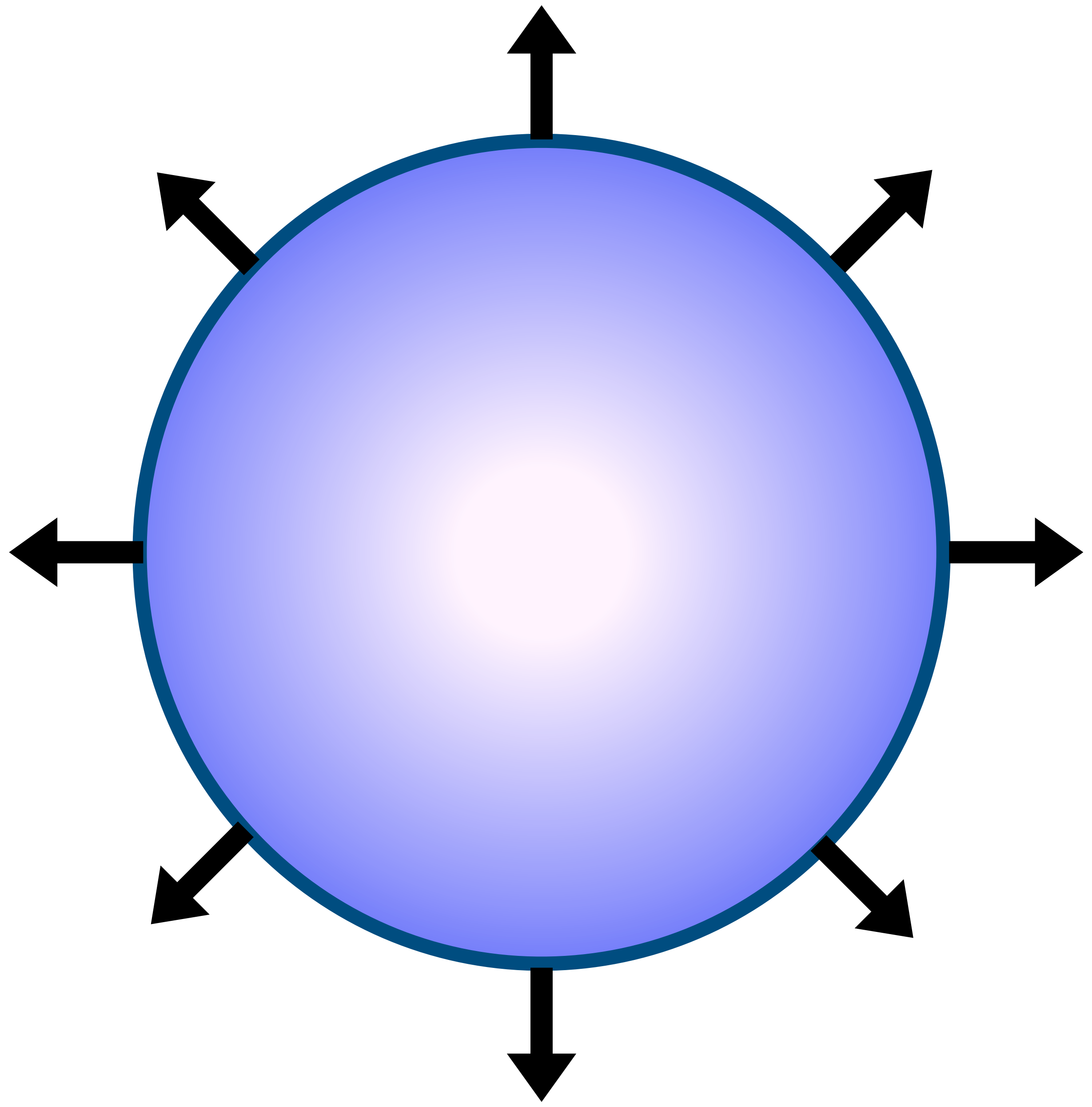

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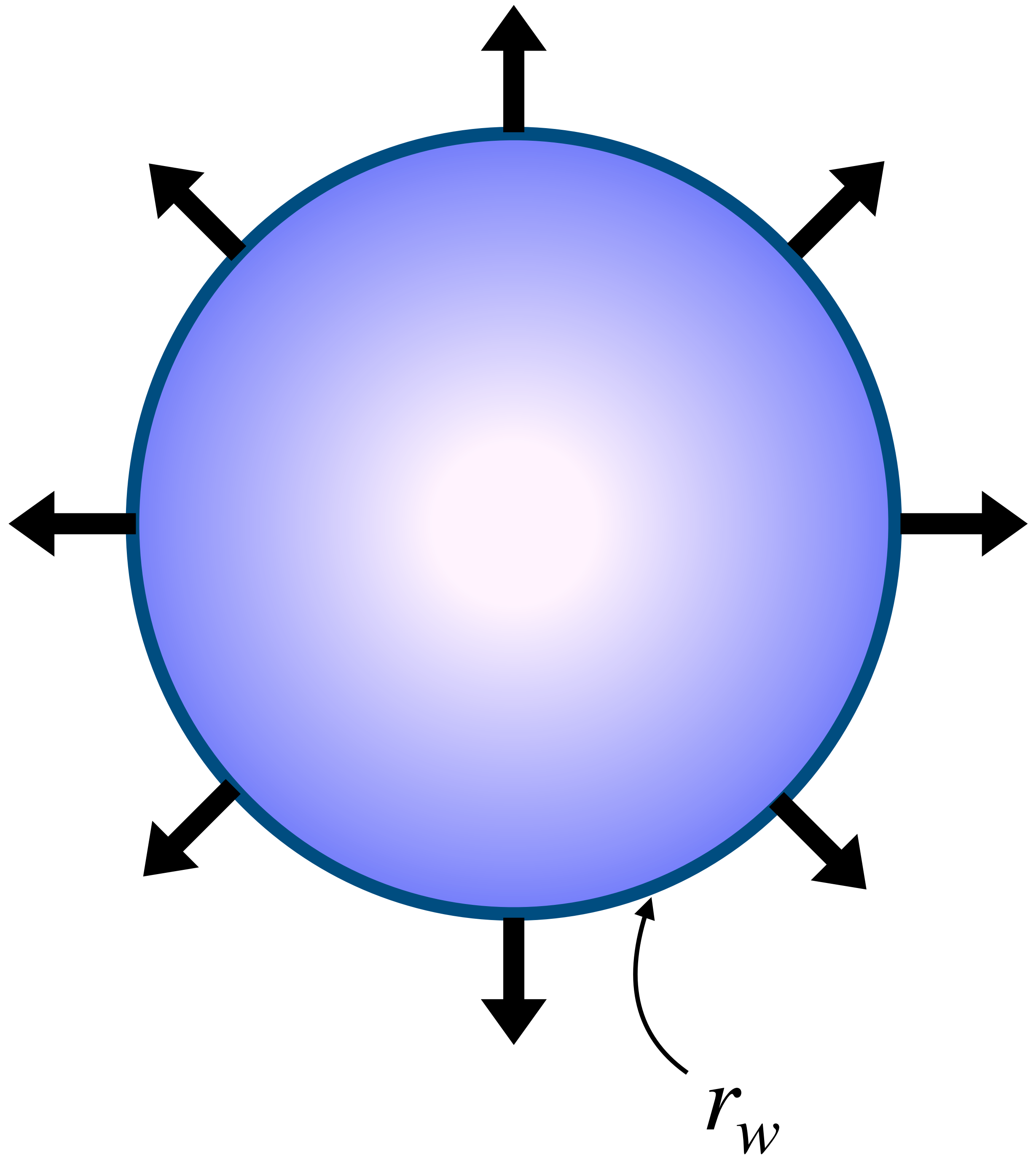

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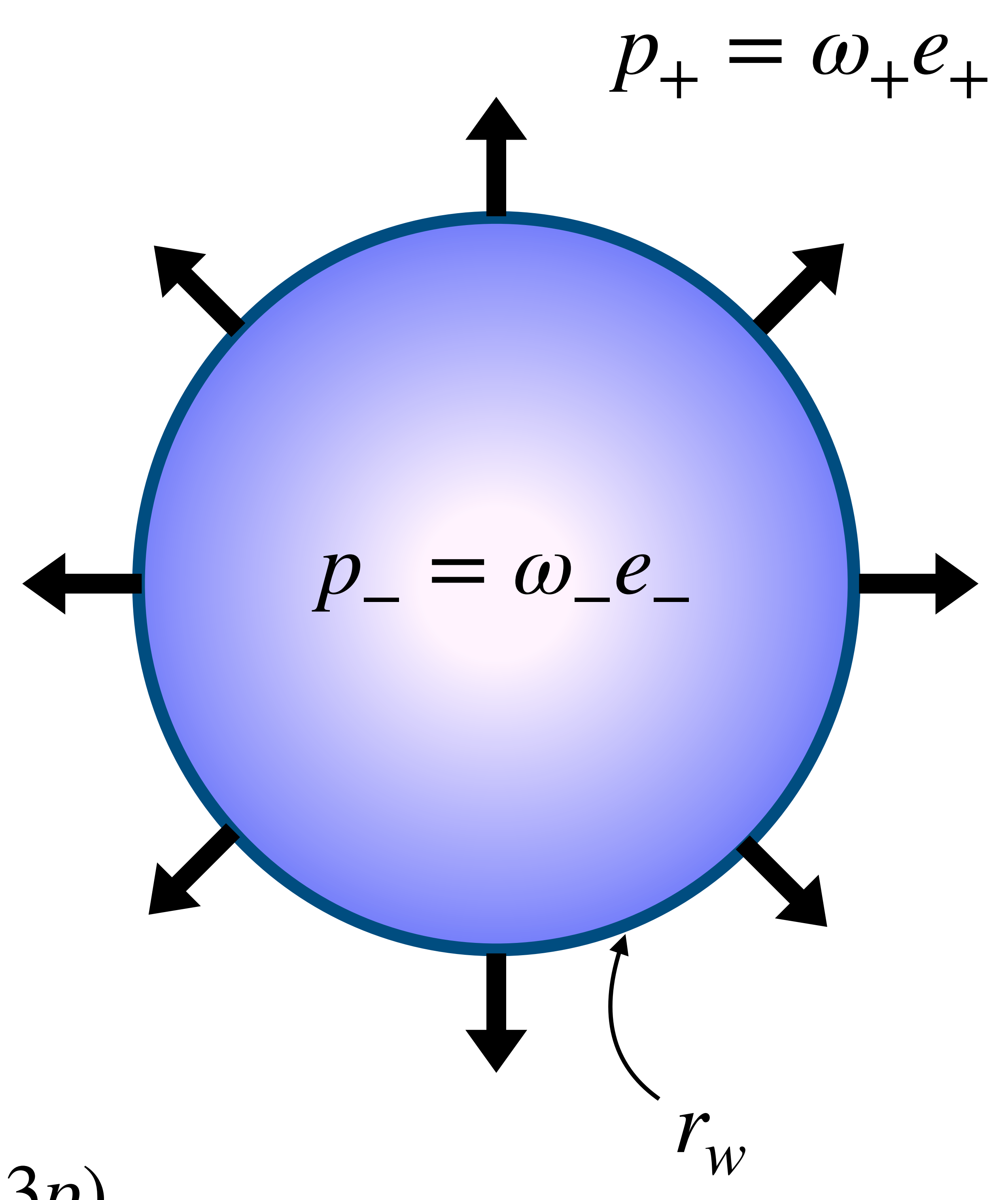
- Equation of state: $p = \omega e, \quad \omega = \omega_- \Theta(r_w(t) - r) + \omega_+ \Theta(r - r_w(t))$

Strength parameter: at the wall $\alpha_+ = \frac{4}{3}\frac{\theta_+ - \theta_-}{w_+}$ Trace anomaly $\theta = \frac{1}{3}(e + 3p)$



Hydrodynamic description of a single expanding bubble

The profile of the bubble is given by the solution of the system of Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$

$$\frac{d\ln U}{d\ln\xi} = \left[(\Phi+\omega\Omega)^2 - 2c_s^2\Gamma^2\Phi\right]\left[\frac{\Omega-\Phi}{U^2(\Phi+\omega\Omega)^2 - c_s^2\Gamma^2(\Omega-\Phi)^2}\right],$$

$$\frac{d\ln\Omega}{d\ln\xi} = \frac{\Omega-\Phi}{\Phi+\omega\Omega}\left[2\omega + (1+\omega)\frac{d\ln U}{d\ln\xi}\right],$$

$$\frac{d\ln\Phi}{d\ln\xi} = \frac{1}{\Phi}(\Omega-\Phi)\,.$$

$$a = \frac{\Omega-\Phi}{(1+\omega)U\Omega}\xi$$

$U \equiv \frac{1}{a}\partial_t R$ — Radial fluid 4-velocity Eulerian observer

$\Gamma \equiv \frac{1}{b}\partial_r R$ — Generalised Lorentz Gamma factor

$\Omega \equiv 4\pi e R^2$ — Energy on a shell of radius R

$\Phi \equiv \frac{M}{R}$ — Gravitational potential at radius R

$$1 - 2\frac{M}{R} \equiv \partial_\mu R \partial^\mu R = \Gamma^2 - U$$

Misner & Sharp (1964) Phys.Rev 136 B571

Asymptotic solutions

$\xi \to 0$

One parameter family of solutions

$$U(\xi \to 0) = \frac{2}{3(1+\omega_-)}\xi$$

$$\Omega(\xi \to 0) = 3k\xi^2$$

$$\Phi(\xi \to 0) = k\xi^2$$

Spatial curvature at the origin

$$R_0^{(3)} = R^{(3)}(\xi \to 0) = \frac{12\xi^2}{R^2}\left[k - \frac{2}{9(1+\omega_-)^2}\right]$$

Since we expect lower energy density in the interior

$$0 < k < \frac{2}{9(1+\omega)^2}$$

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$\xi \to \infty$

Flat FLRW solution

$$U_F = \frac{2}{3(1+\omega_+)}\frac{\xi}{a_F} \qquad U_F^2 = 2\Phi_F \qquad \Omega_F = 3\Phi_F$$

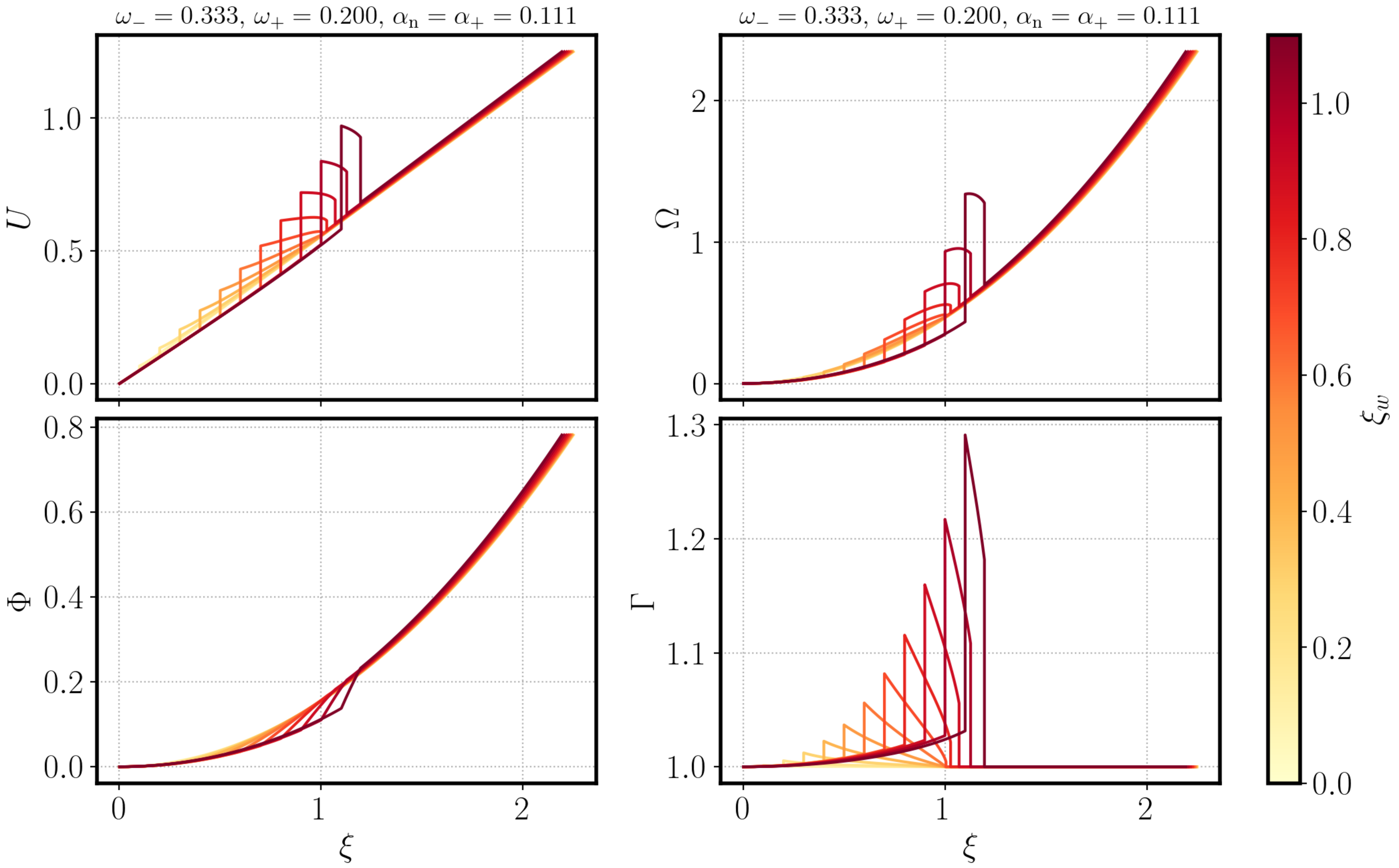
Constant-ξ observers: $V_\xi^\mu = \gamma\left(\frac{1}{a},\frac{v}{b},0,0\right) \quad v = \frac{\xi - aU}{a\Gamma}$

$$u = \frac{v_F - v}{1 - v v_F}$$

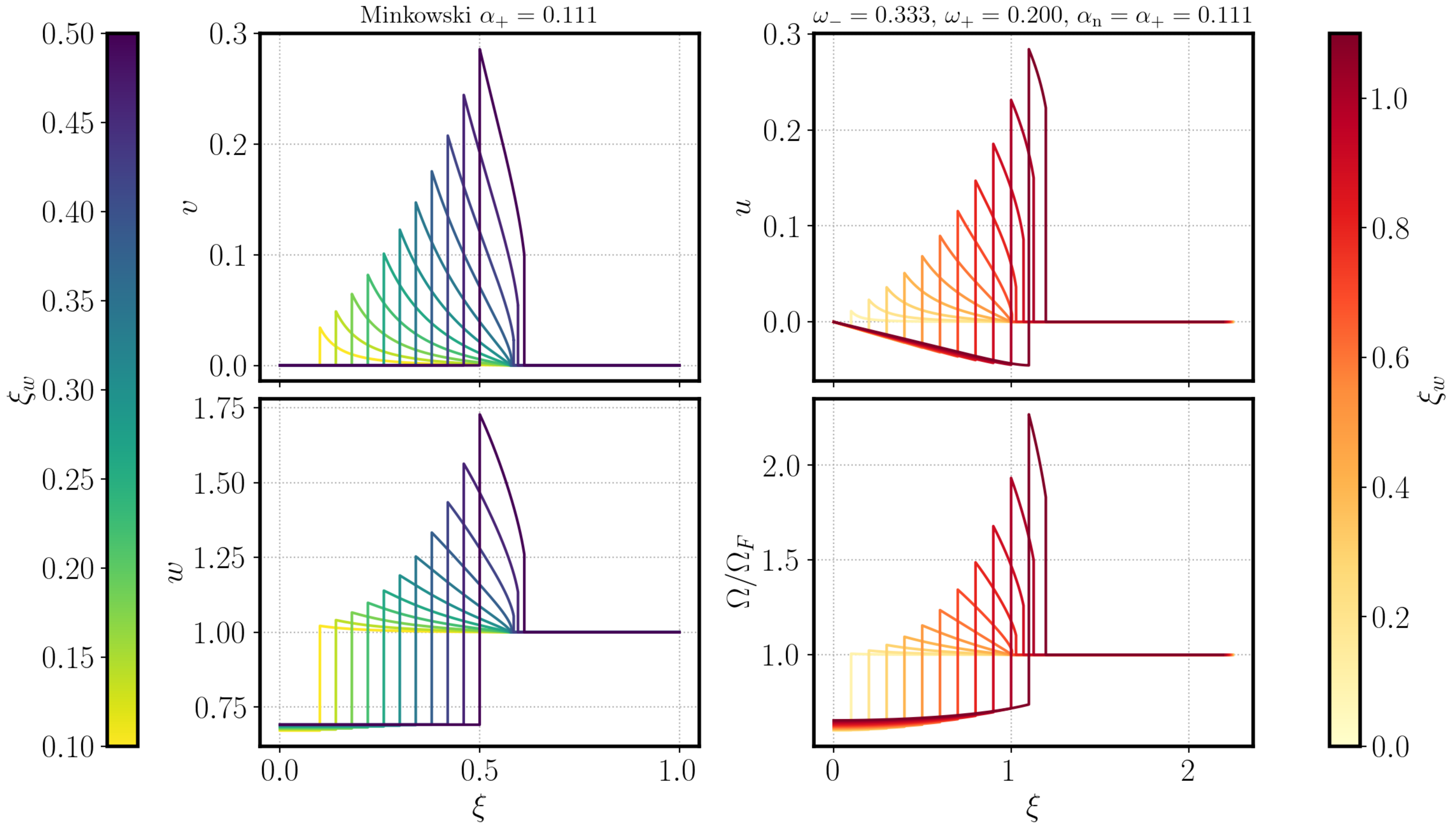
Relative velocity between V_ξ^μ and another hypothetical constant-ξ observer that lives at the same ξ in FLRW

Measure departure from FLRW

Deflagration solutions: $v(\xi_w)_- < c_{s-}$



Deflagration solutions: $v(\xi_w)_- < c_{s-}$ - Comparison with Minkowski



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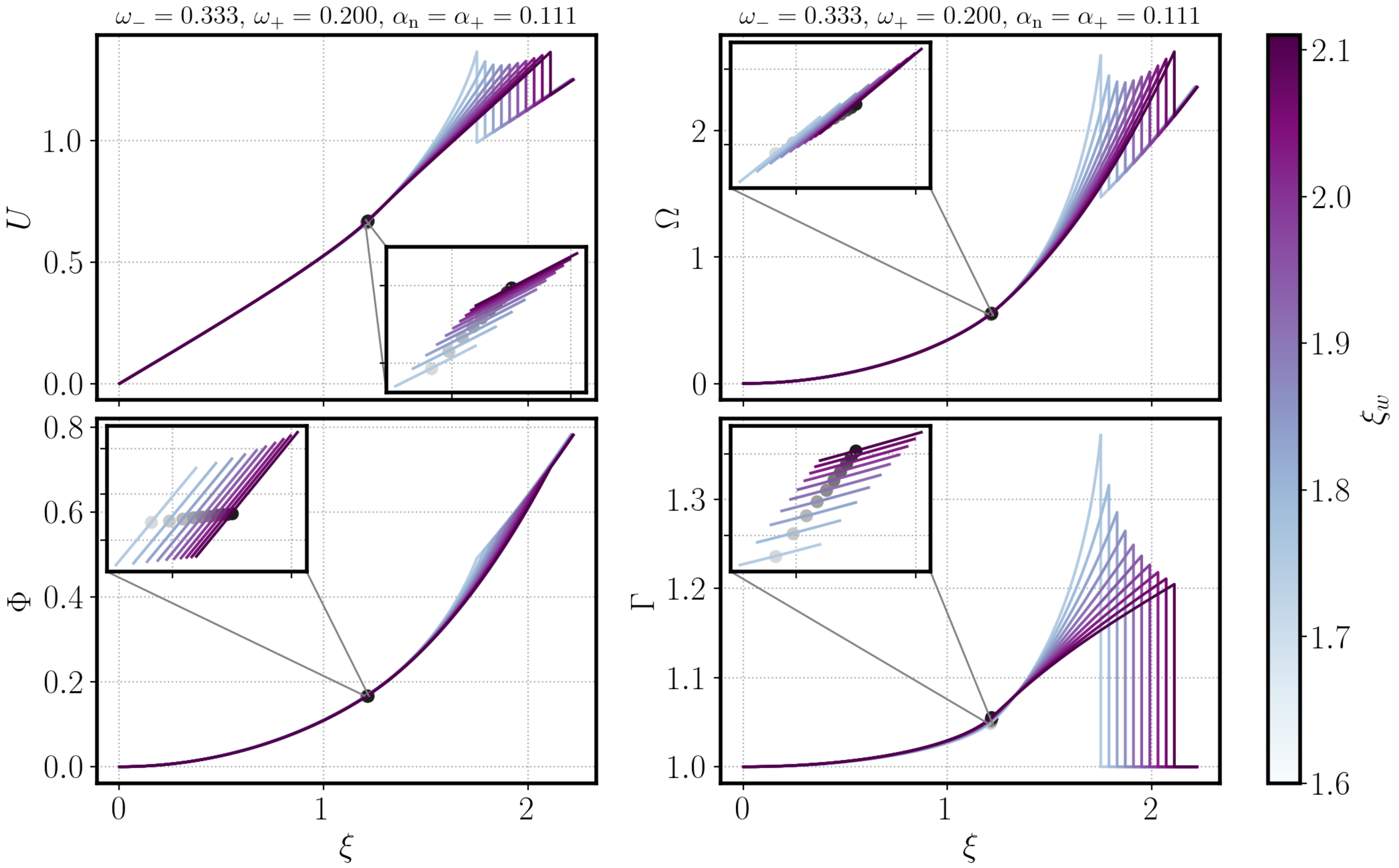

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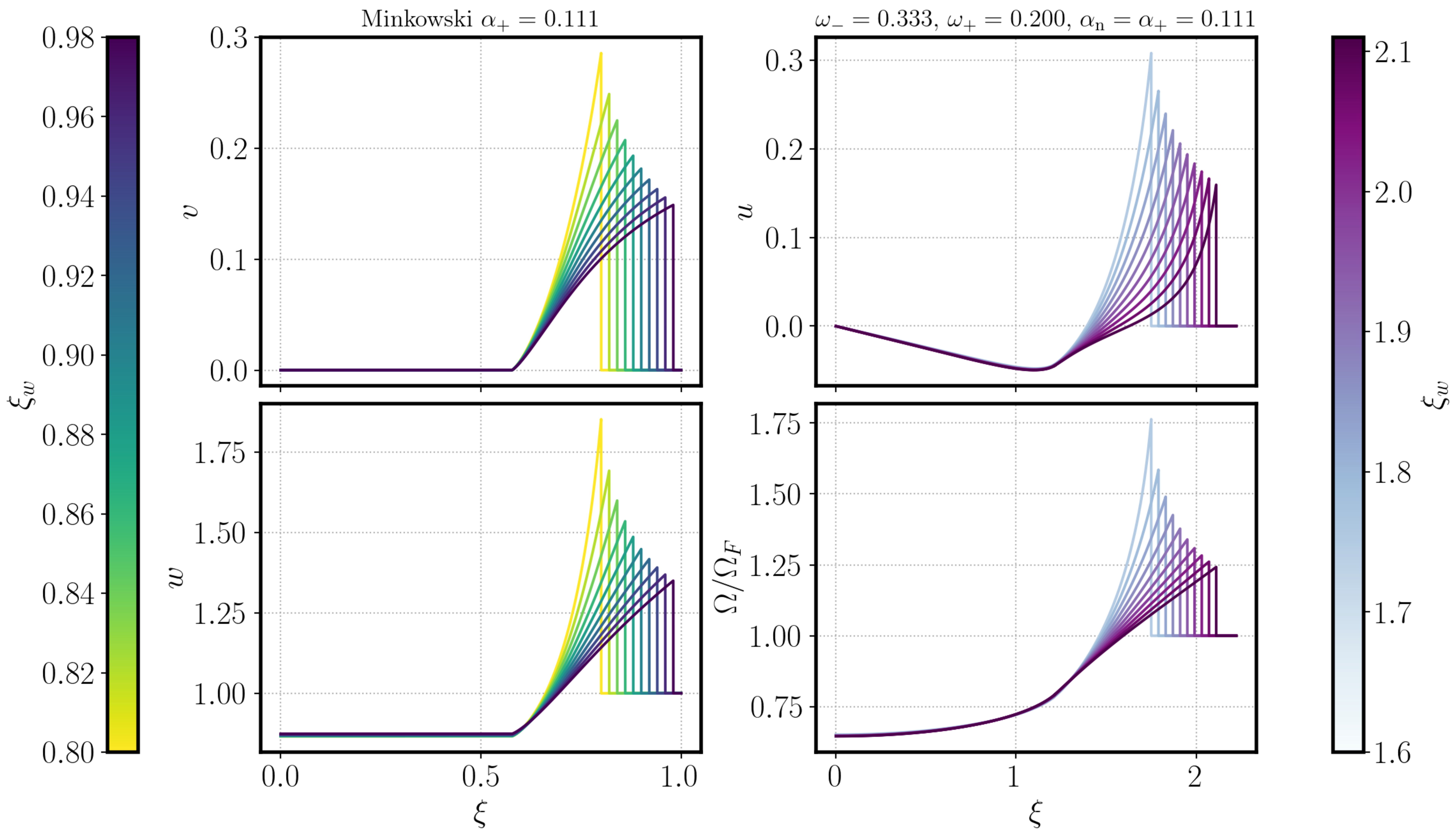


Curvature of spatial sections around the origin

Projection tensor

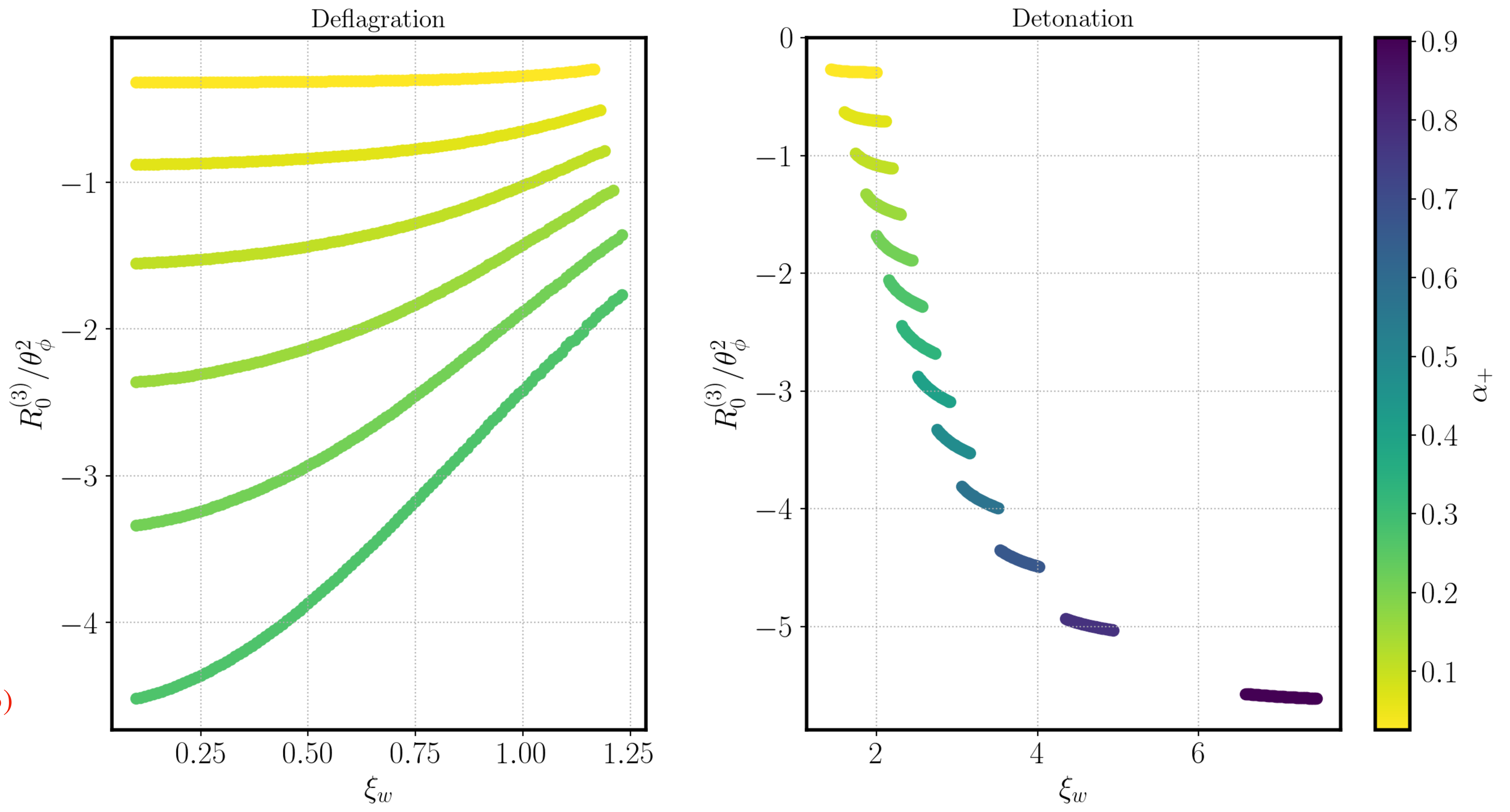
$$\tilde{h}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$

Expansion tensor

$$\theta_{\mu\nu} = \tilde{h}^{\alpha}{}_{\mu}\tilde{h}^{\beta}{}_{\nu}\nabla_\alpha u_\beta$$

$$\theta_\phi \equiv \theta^\phi_\phi$$

L. Rezzolla and O. Zanotti (2013)



Conclusions & future developments

- Detonation and deflagration solution exist in GR ⟶ Hybrids
- Detonations and deflagrations have peculiar rarefaction waves
- Spatial sections around the origin are negatively curved (but it's not a FLRW Universe)
- The amount of curvature in the interior is significantly larger than naive expectation

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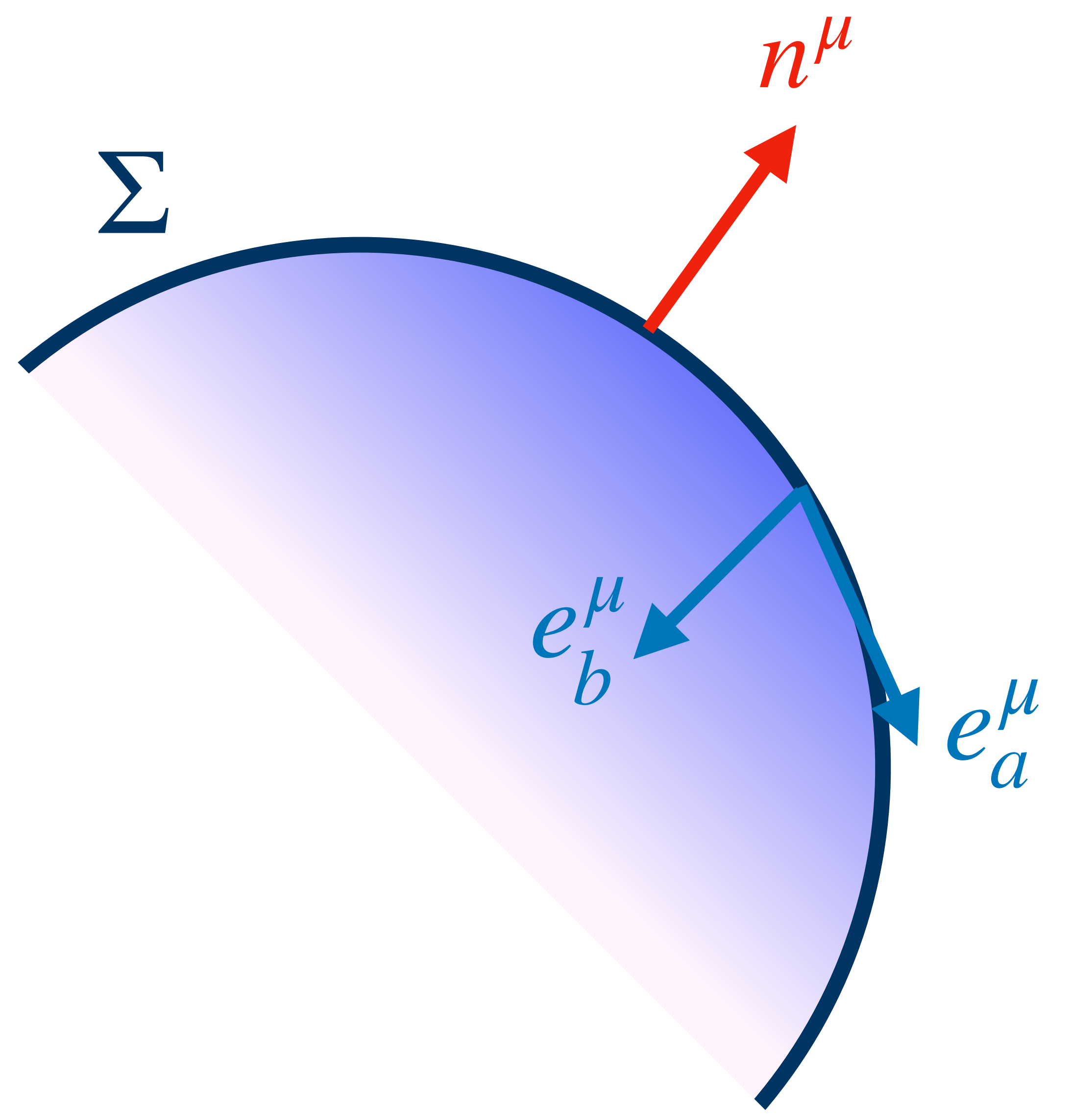
  ⤷ Calculation of secondary GW?

Backup slides

Junction conditions

Introduce a tetrad: $\left\{ n^{\mu}, e^{\mu}{}_{\tau}, e^{\mu}{}_{\theta}, e^{\mu}{}_{\phi} \right\}$



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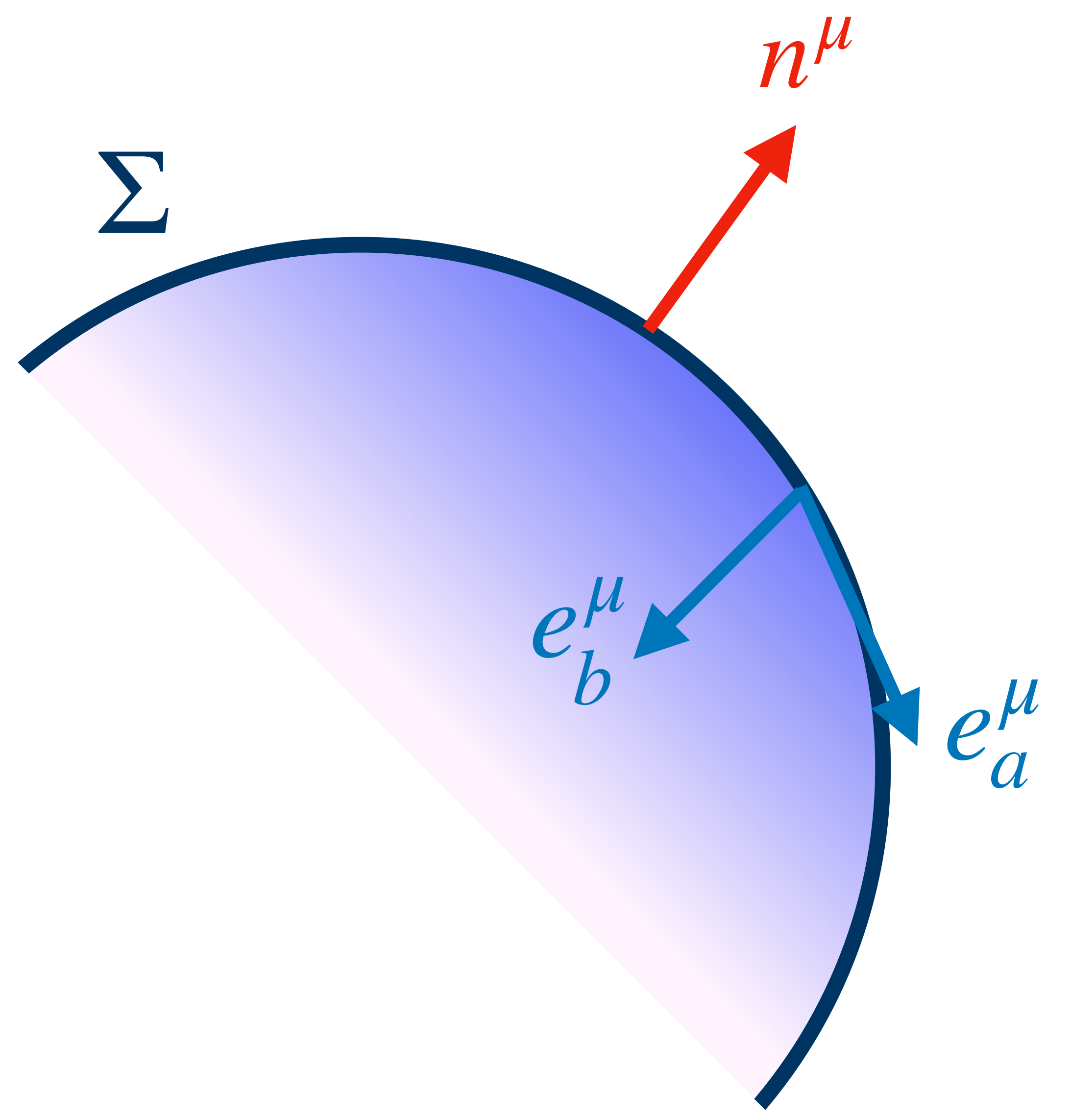

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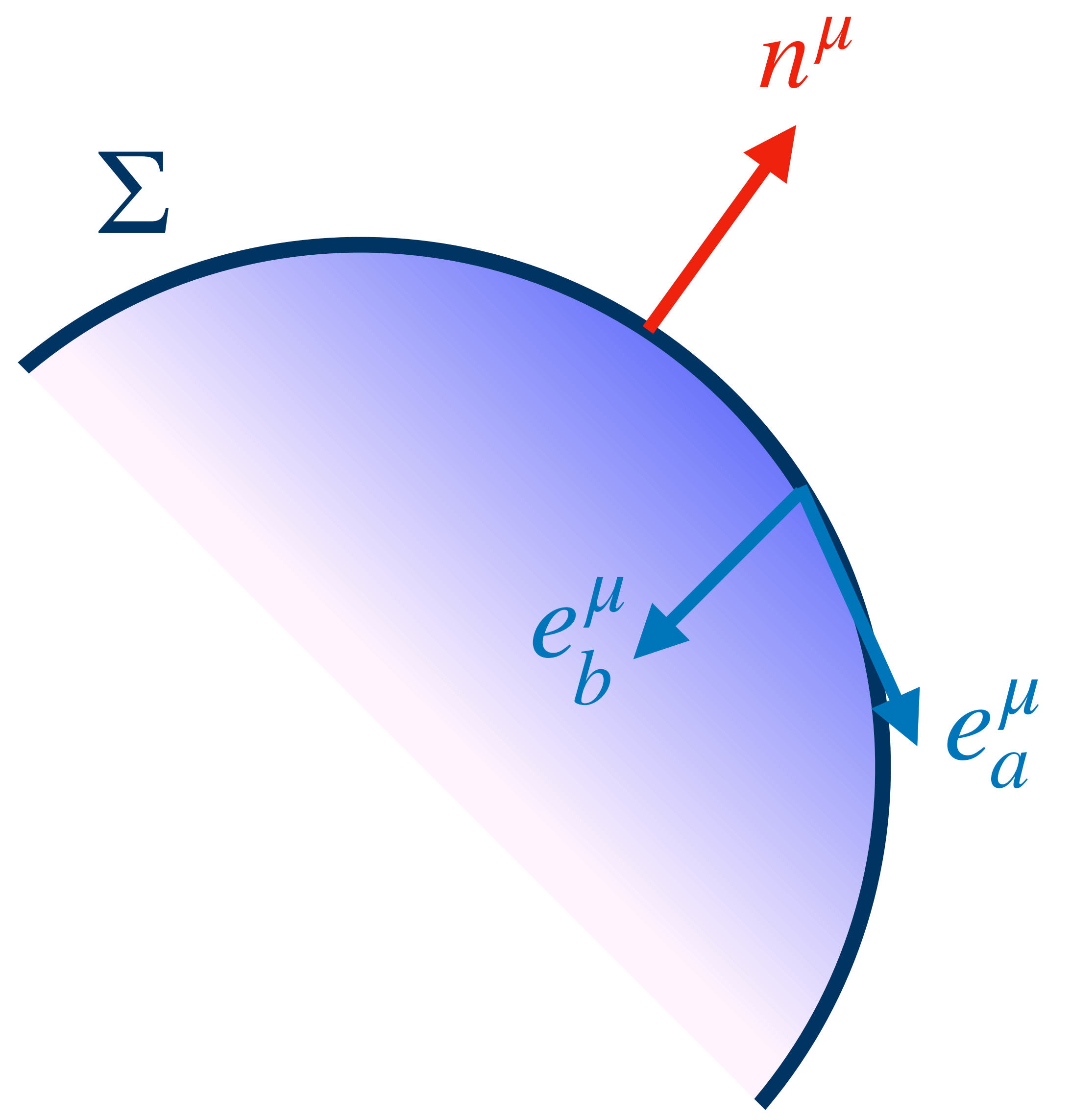

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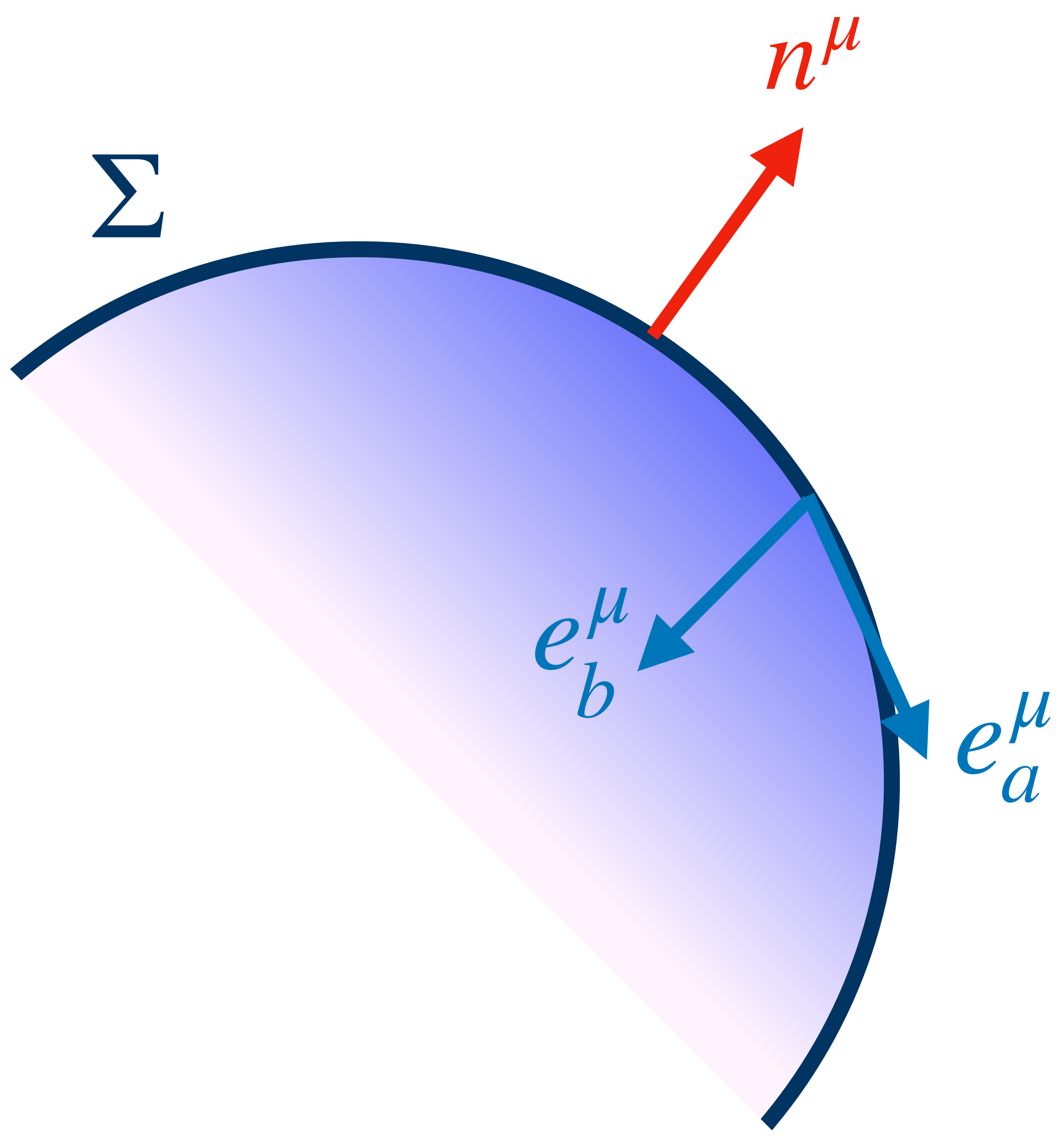
Israel junction conditions:

$$[h_{ab}]^\pm = 0$$

$$[K_{ab}]^\pm = -8\pi\left(S_{ab} - \frac{1}{2} g_{ab} S\right)$$



$S^a{}_b$: surface stress energy tensor

$$[A]^\pm = A_+ - A_-$$
$$\{A\}^\pm = A_+ + A_-$$

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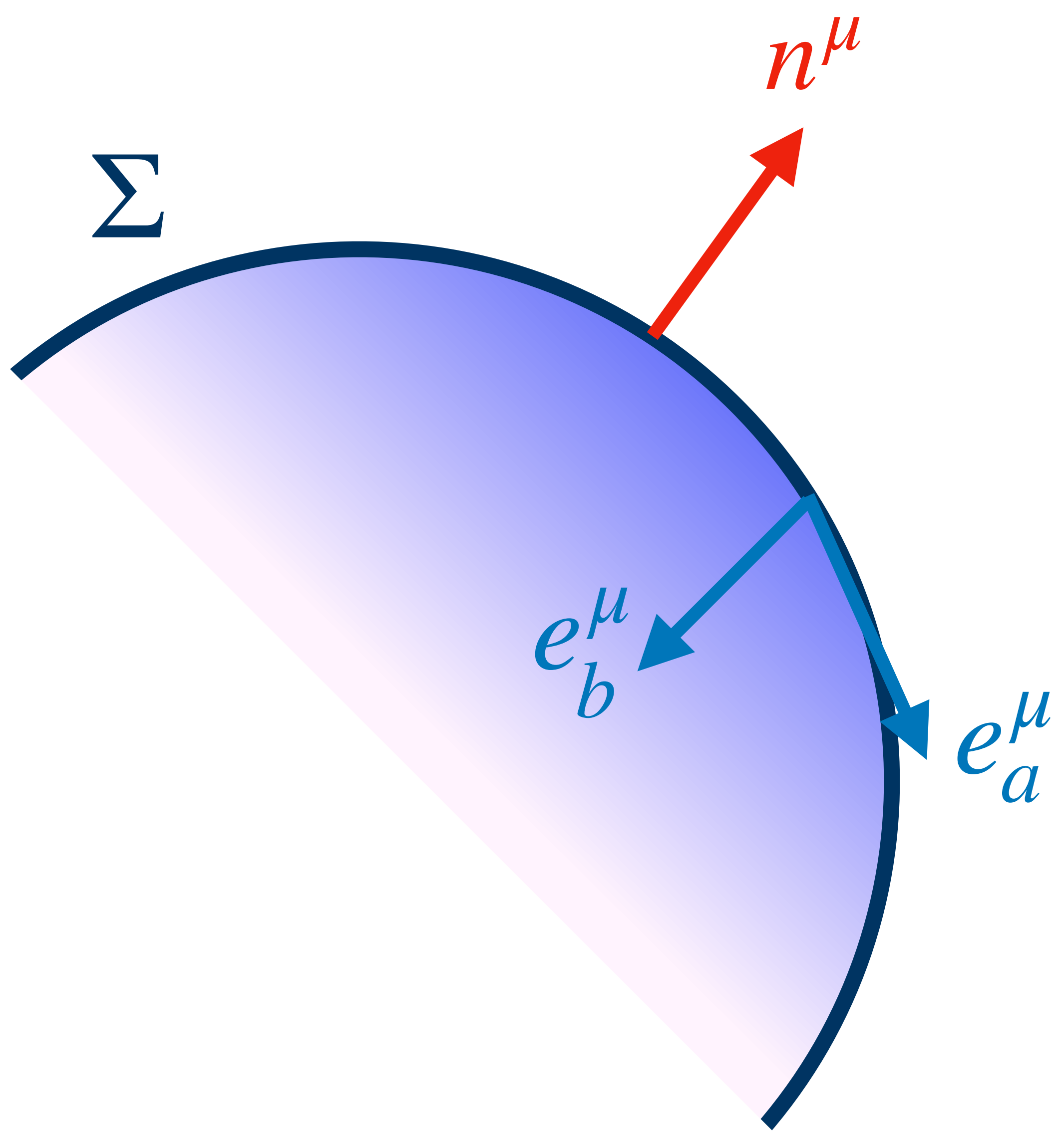

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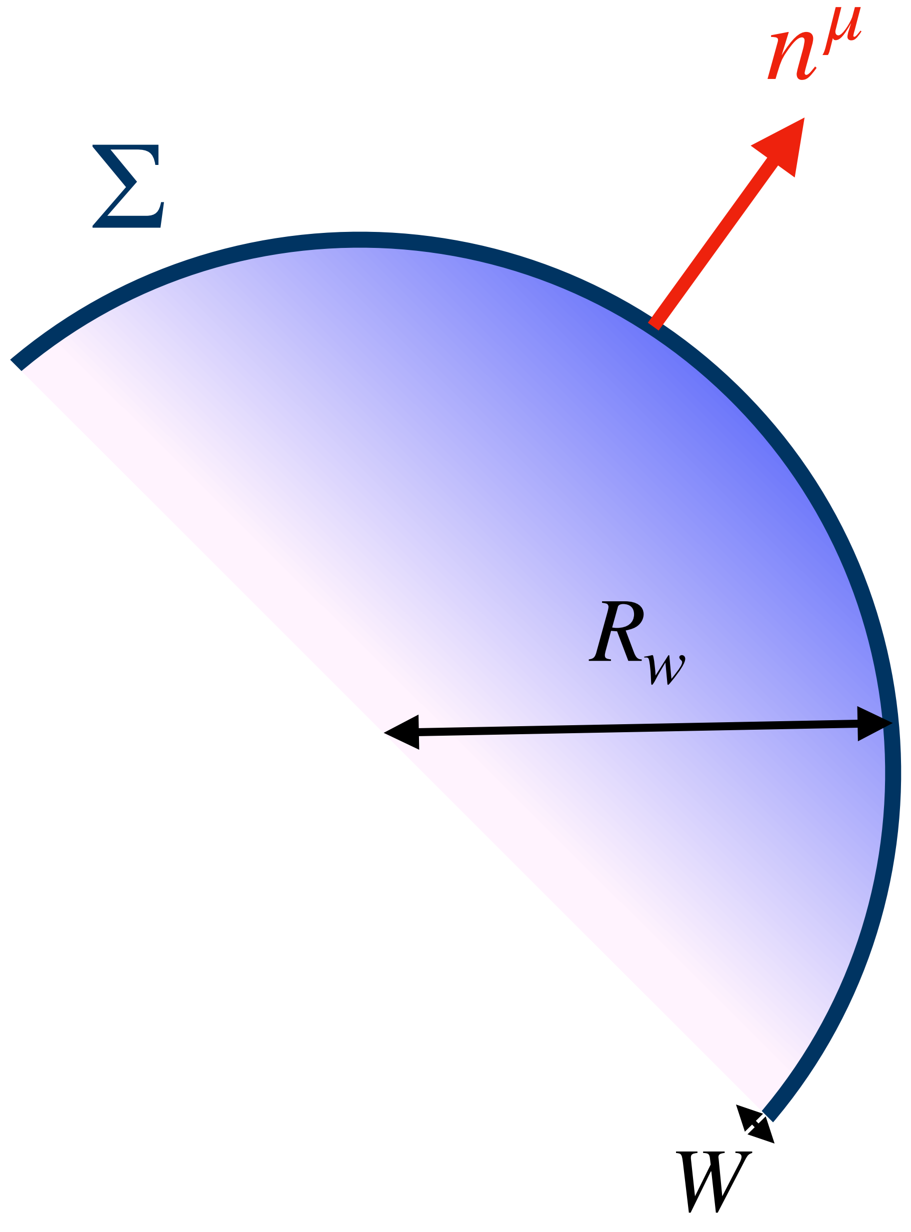
Dimensional analysis estimate $\mathscr{D}, K \sim R_w^{-1}, \qquad S \sim \bar{e}W \sim T_c^4 W$



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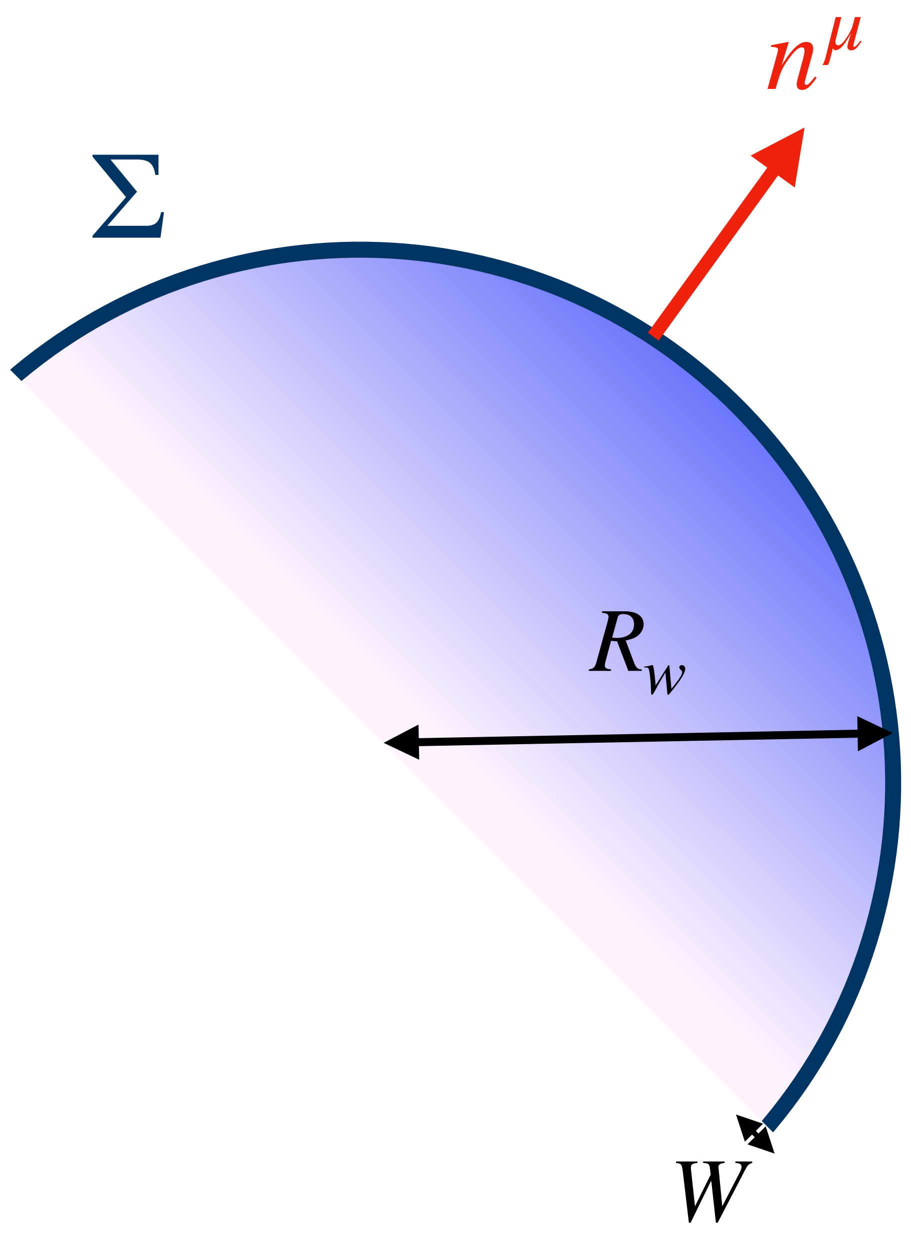

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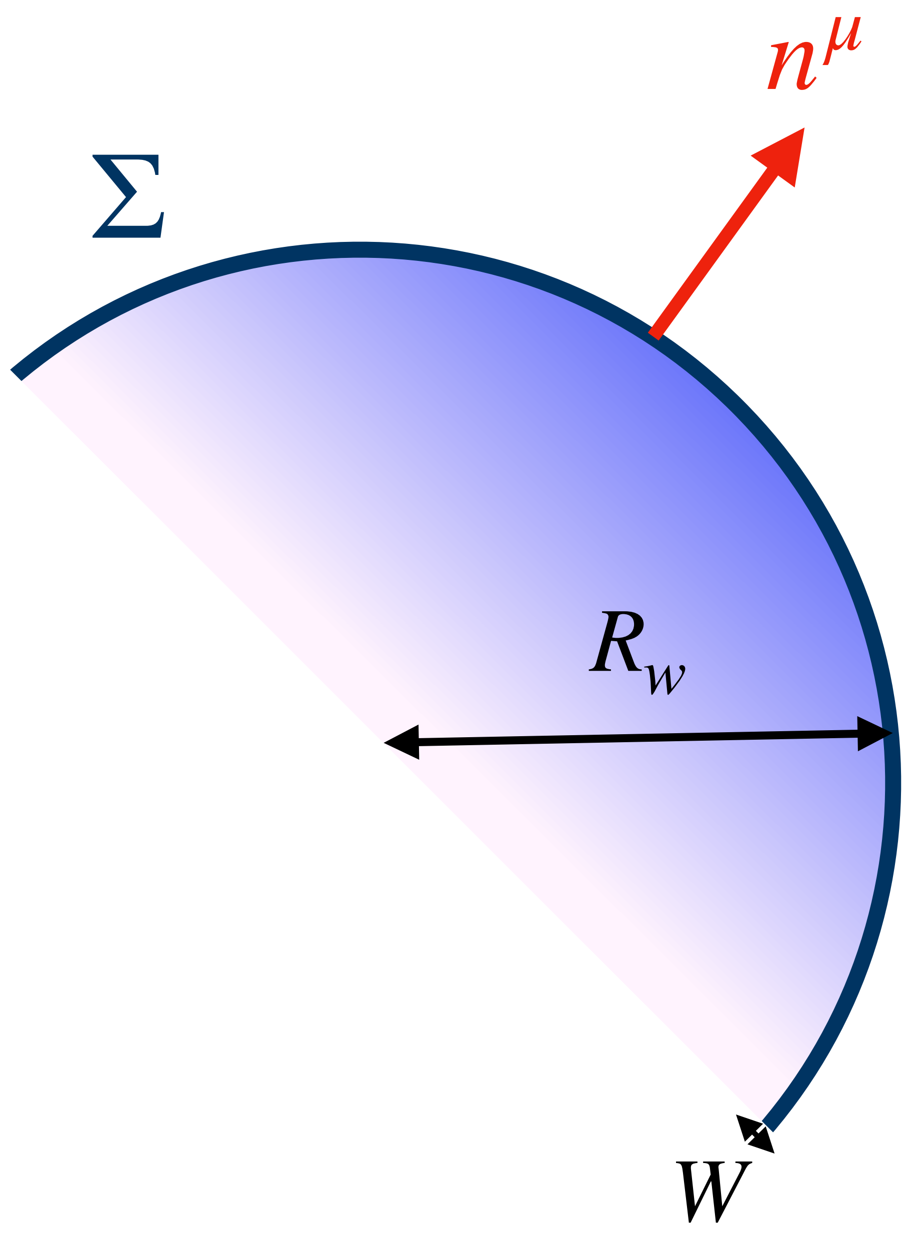

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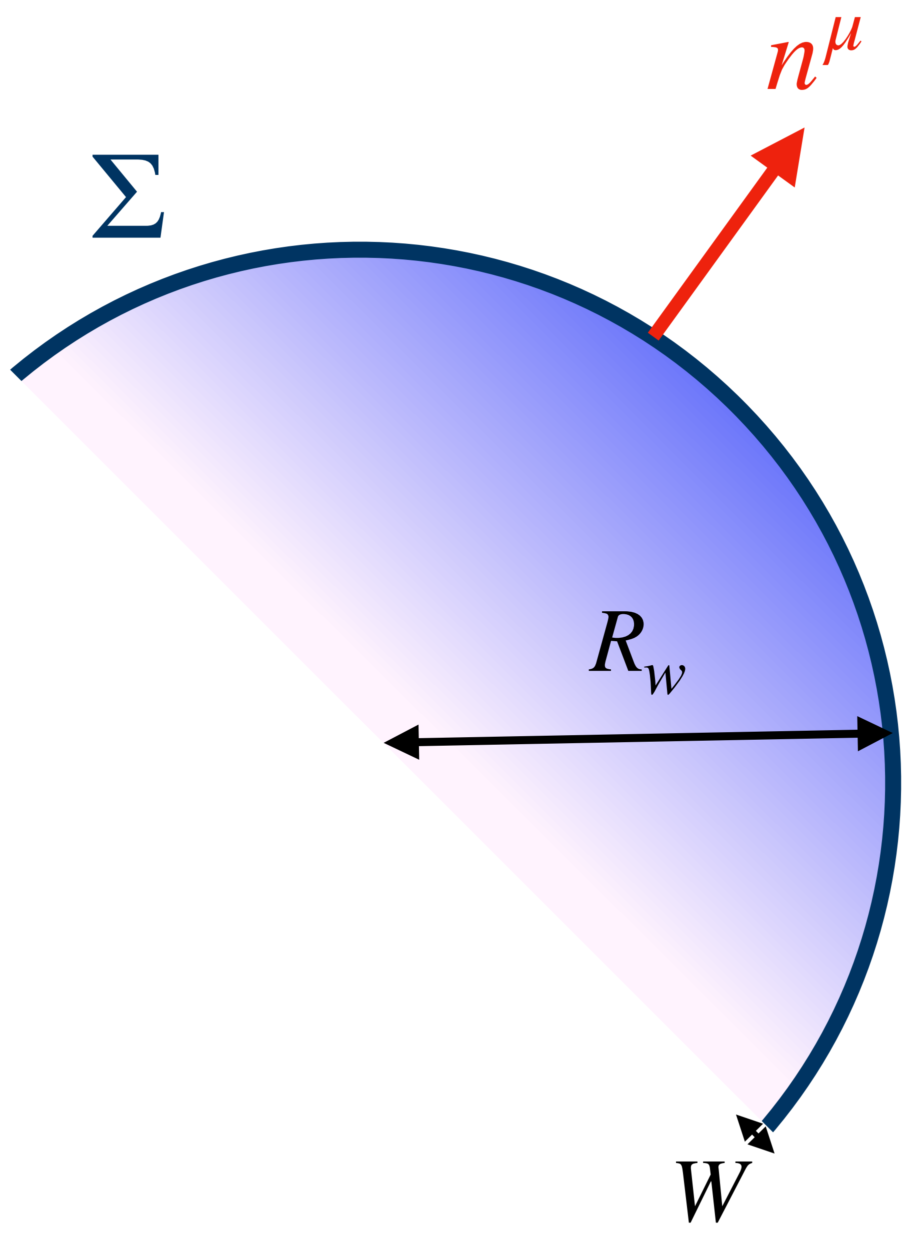

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$$\left[T_{\mu\nu} n^{\mu} n^{\nu}\right]^{\pm} = 0$$

$\longrightarrow$

$$[R]^{\pm} = 0$$
$$\left[\Phi\right]^{\pm} = 0$$
$$\left[(1+\omega)\Omega v \gamma^2\right]^{\pm} = 0$$
$$\left[(1+\omega)\Omega v^2 \gamma^2 + \omega\Omega\right]^{\pm} = 0$$

Fixed points

i. $(U, \Omega, \Phi) = (0,0,0)$

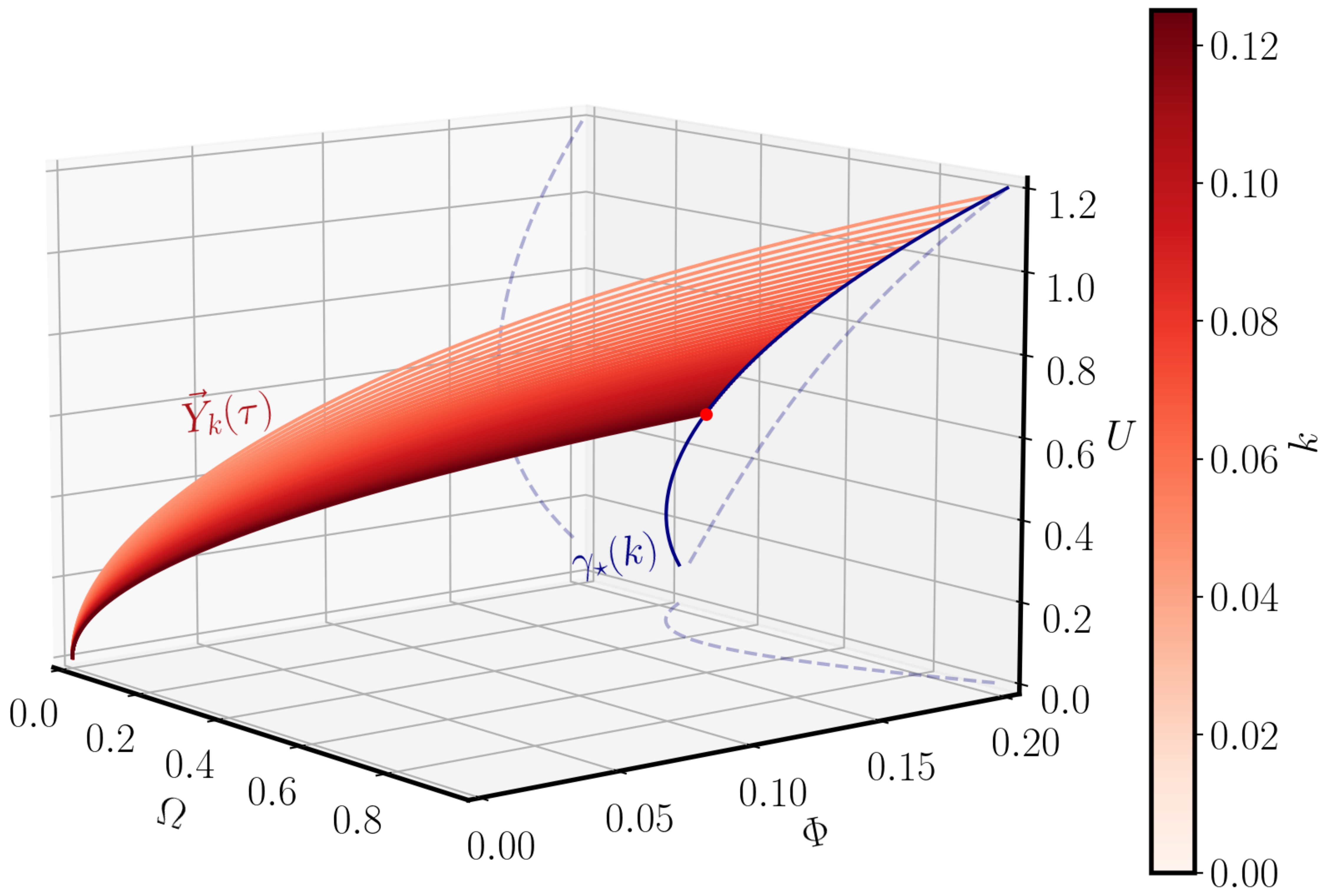
ii. $U = 0, \quad \Omega = \Phi$

iii. $c_{s_-} = \frac{U_\star}{\Gamma_\star} \frac{\Phi_\star + \omega\Omega_\star}{\Omega_\star - \Phi_\star}, \qquad U_\star = \frac{\Omega_\star - \Phi_\star}{\sqrt{2\Phi_\star}}$

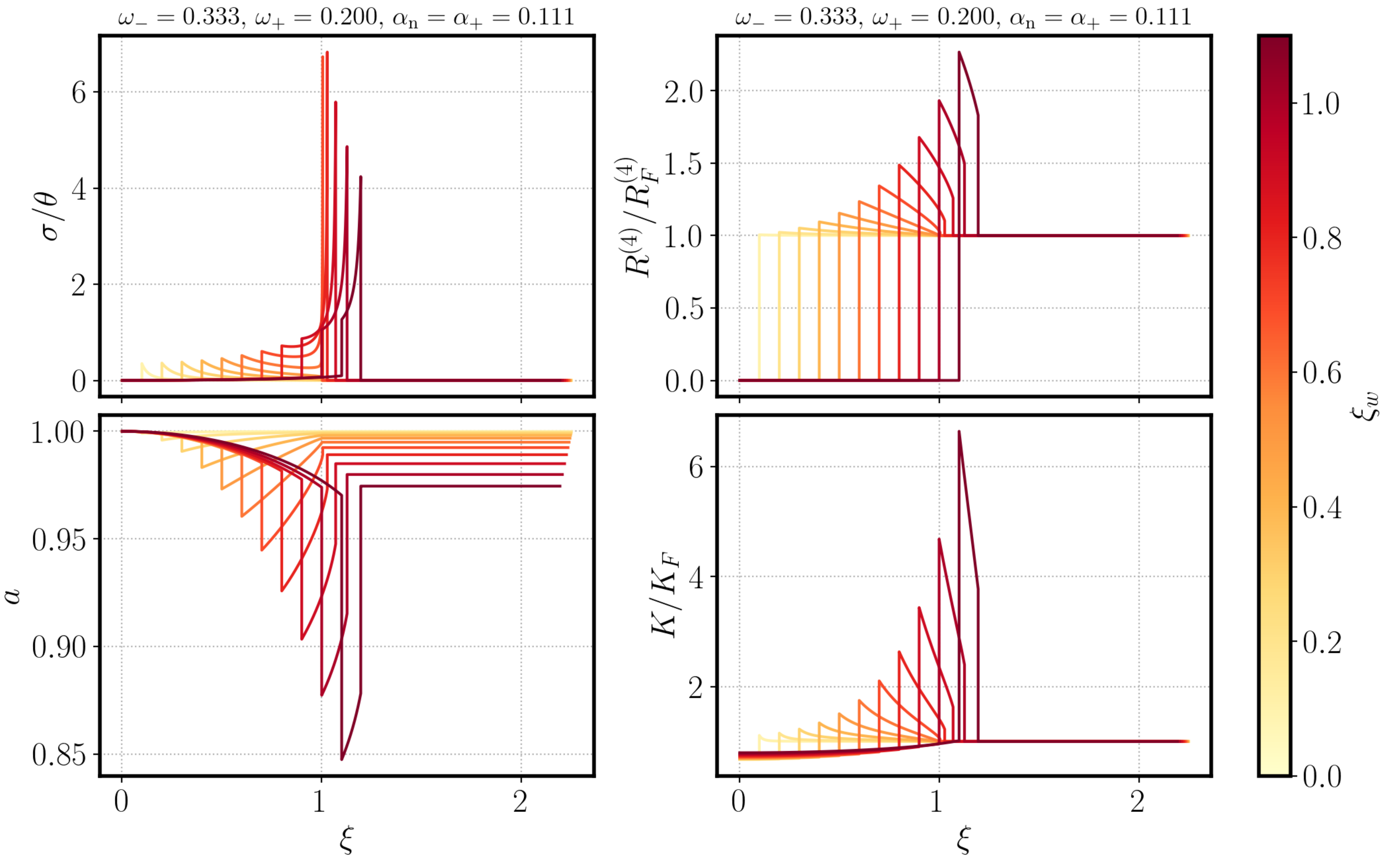
$\overrightarrow{Y}_k = (U_k, \Omega_k, \Phi_k)$ Trajectory of solutions of the Einstein equations with initial condition k

The endpoint of $\overrightarrow{Y}_k = (U_k, \Omega_k, \Phi_k)$ move along a line of fixed points $\gamma_\star(k) = (\xi_\star(k), \overrightarrow{Y}_\star(k))$

$\xi_\star(k)$ fixed by the condition $a(\xi \to 0) = 1$



Deflagration: curvature



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$$\theta_{\mu\nu} = \tilde{h}^{\alpha}{}_{\mu}\tilde{h}^{\beta}{}_{\nu}\nabla_\alpha n_\beta$$

$$\sigma_{\mu\nu} = \theta_{(\mu\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\theta\tilde{h}_{\mu\nu}$$

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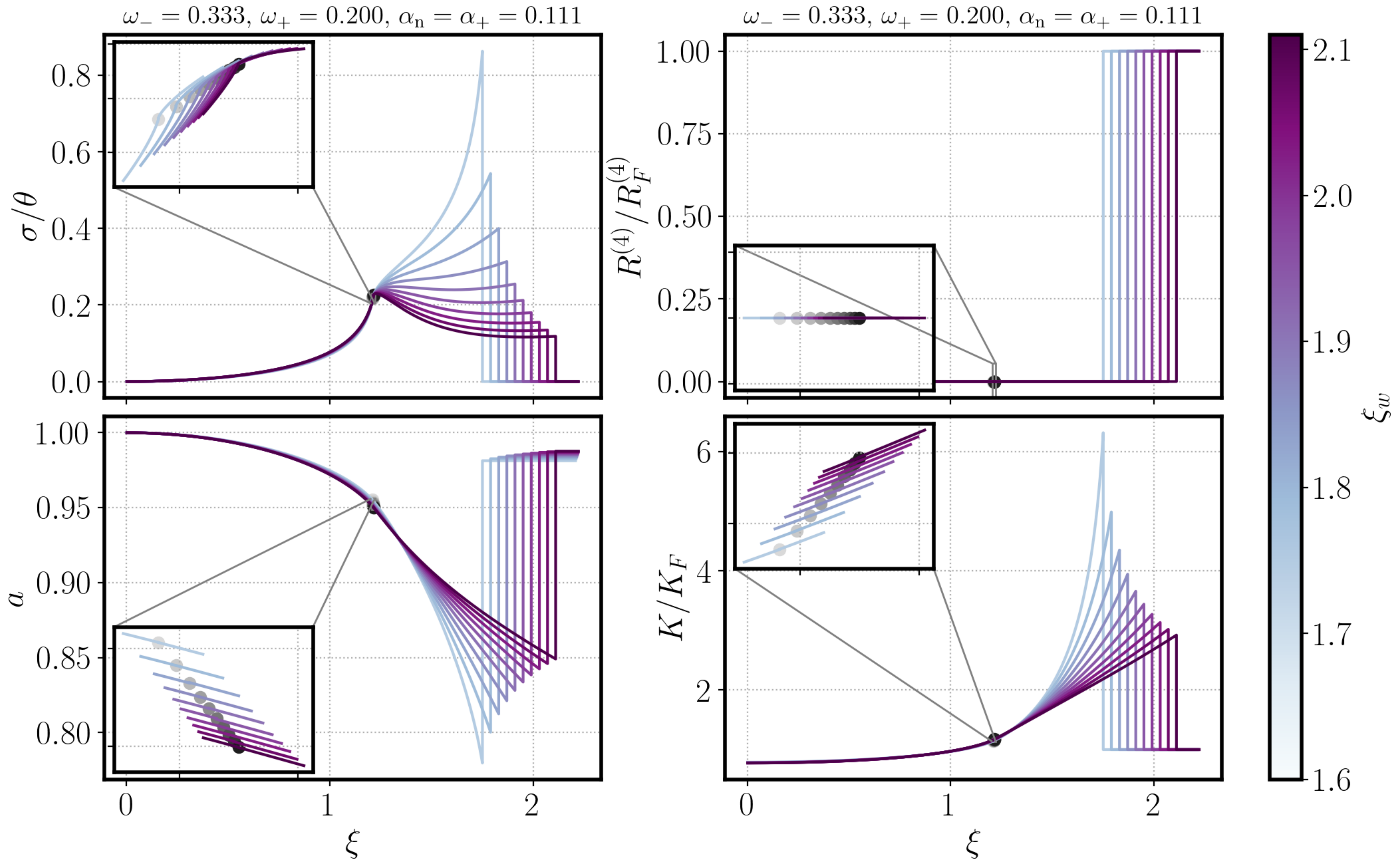

$\tilde{h}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$

$\theta_{\mu\nu} = \tilde{h}^\alpha{}_\mu \tilde{h}^\beta{}_\nu \nabla_\alpha n_\beta$

$\sigma_{\mu\nu} = \theta_{(\mu\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{3}\theta\tilde{h}_{\mu\nu}$