Hard Thermal Loops
Resummations and collective modes

- To see the emergence of collectivity, consider resummed propagators
- Quarks: in vacuum propagating massless quark (antiquark) with positive (negative) chirality to helicity ratio $S_R(Q) = h_q^+ S_R^+(Q) + h_q^- S_R^-(Q)$, with $S_R^\pm(Q) = i/(q^0 + i\epsilon \mp q)$ and $h_q^\pm = (\gamma^0 \mp \gamma \cdot \overrightarrow{q})/2$
- After HTL resummation

$$S_R^\pm(\mathcal{Q}) = \frac{i}{q^0 \mp (q + \Sigma_R^\pm(q^0/q))} = \frac{i}{q^0 \mp \left[q + \frac{m_\infty^2}{2q}\left(1 - \frac{q^0 \mp q}{2q}\ln\left(\frac{q^0+q}{q^0-q}\right)\right)\right]}$$

Hard Thermal Loops

Collective quark modes

$$S_R^{\pm}(Q) = \frac{i}{q^0 \mp (q + \Sigma_R^{\pm}(q^0/q))} = \frac{i}{q^0 \mp \left[q + \frac{m_\infty^2}{2q}\left(1 - \frac{q^0 \mp q}{2q}\ln\left(\frac{q^0+q}{q^0-q}\right)\right)\right]}$$

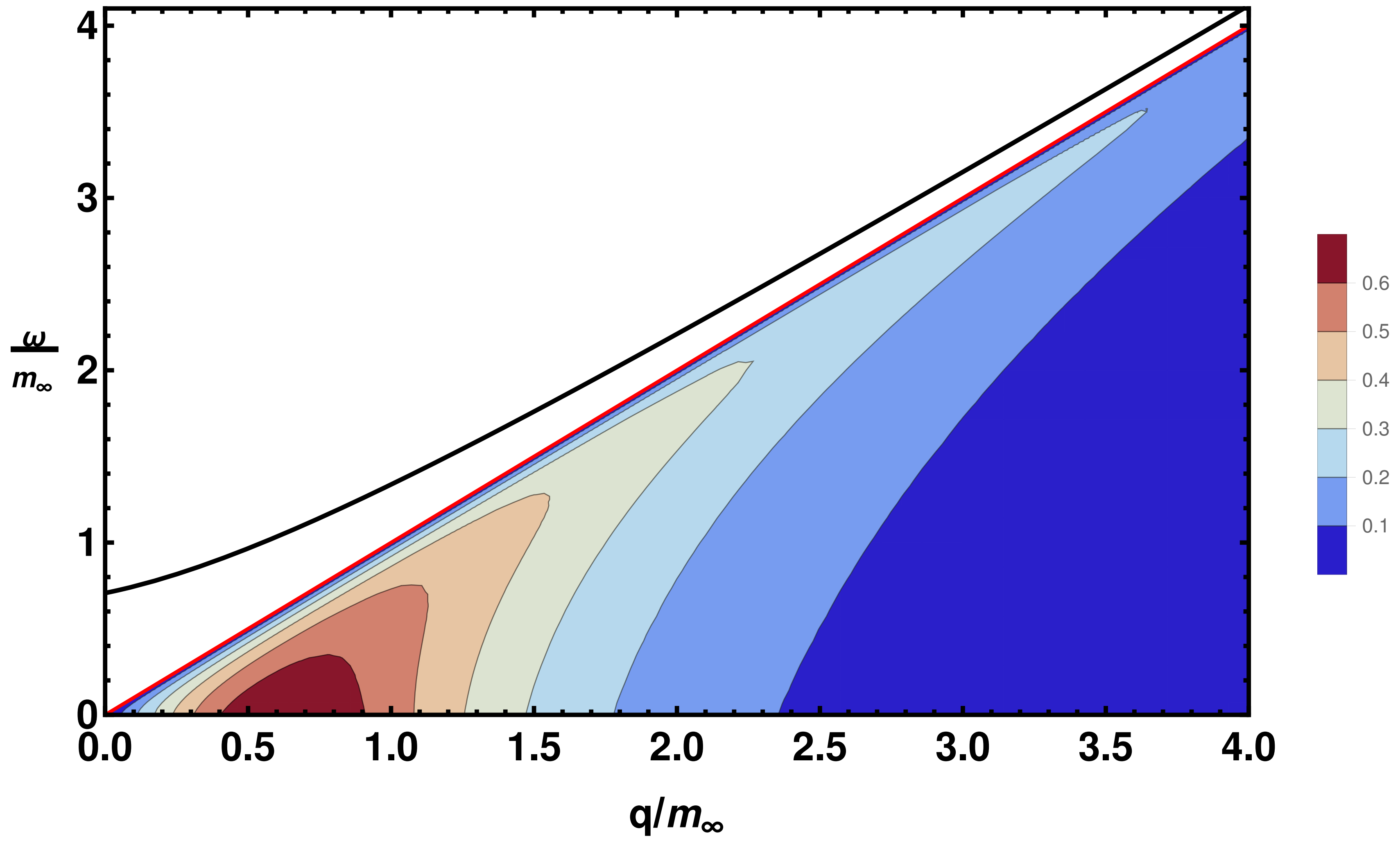
- In the time-like sector plasmons: collective excitations with modified dispersion relation. At vanishing momentum $S_R^{\pm}(q^0) = iq^0/((q_0 + i\epsilon)^2 - m_\infty^2/2)$, propagating, massive modes for both helicities! Plasma oscillations

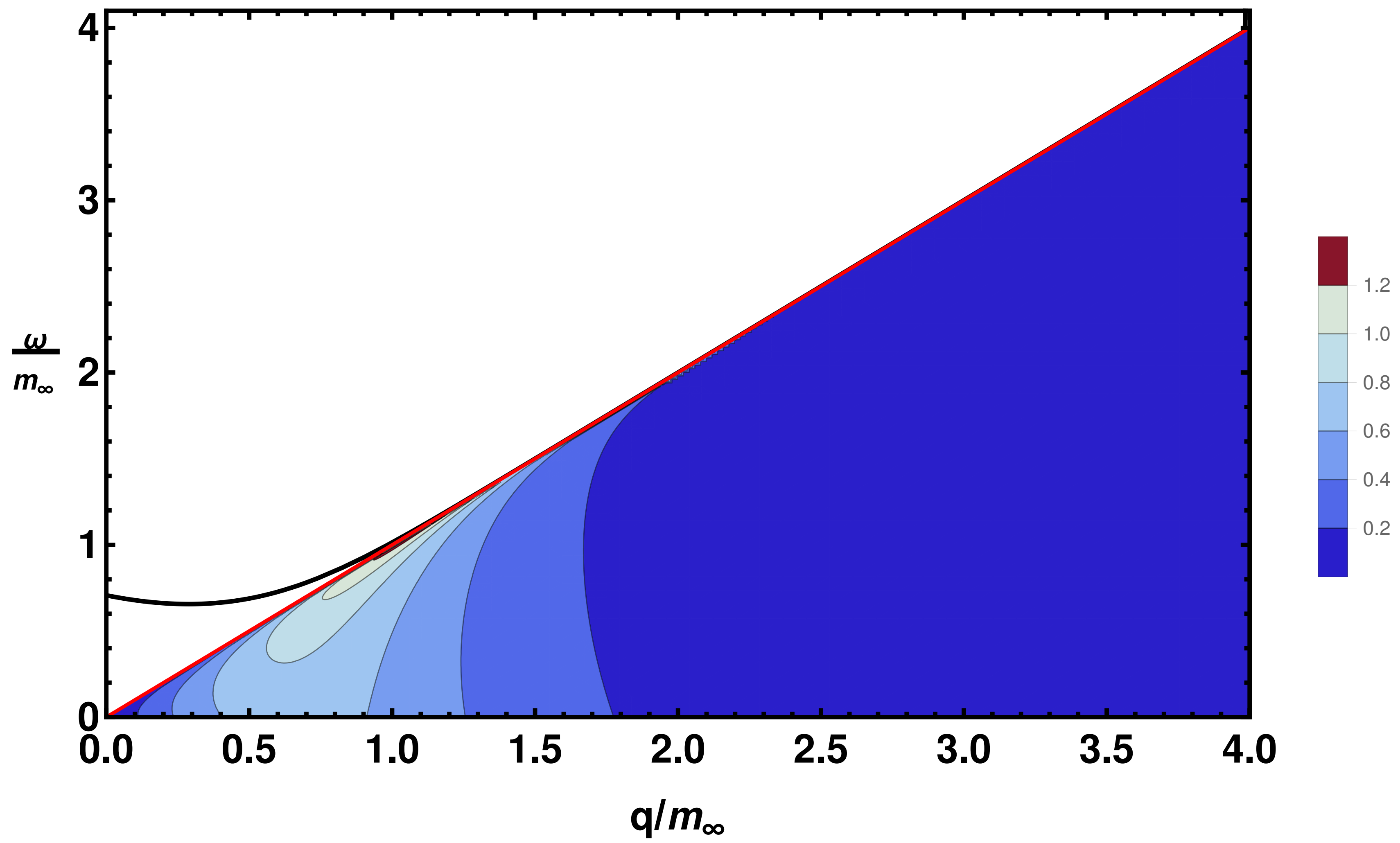
  At large $q \gg m_\infty$, at positive frequency the negative chirality/helicity mode has exponentially small residue, positive chirality/helicity modes have $q^0 = q + m_\infty^2/(2q)$ asymptotic mass. The opposite happens at neg freq. In-between: numerical solution, plasmino

Hard Thermal Loops

Collective quark modes

$$S_R^{\pm}(\mathcal{Q}) = \frac{i}{q^0 \mp (q + \Sigma_R^{\pm}(q^0/q))} = \frac{i}{q^0 \mp \left[q + \frac{m_\infty^2}{2q}\left(1 - \frac{q^0 \mp q}{2q}\ln\left(\frac{q^0+q}{q^0-q}\right)\right)\right]}$$





- Plasmino: a collective excitation where the positive frequency fermion mixes with the negative frequency anti-fermion. Negative freq. derivative!

Hard Thermal Loops

Causality and “old” sum rules

- The analytical properties of amplitudes are dictated by causality.
- They can be used to obtain *sum rules*: perform complicated-looking integrals analytically
- Textbook causality examples: retarded (advanced) propagator analytical in the upper (lower) half-plane in the complex frequency
- “Old” sum rules for HTL propagators at fixed three-momentum exploit it

$$I_E \equiv \frac{1}{d_A}\int d^3\mathbf{x}\, e^{-i\boldsymbol{q}\cdot\mathbf{x}}\langle E^{i\,a}(t=0,\mathbf{x})E^{i\,a}(0,\mathbf{0})\rangle = \int \frac{d\omega}{2\pi}\,\frac{T}{\omega}\left[2\omega^2\rho_T(\omega,q)+q^2\rho_L(\omega,q)\right]$$

- Aside: HTLs are classical (see Kurkela), kept only the classical-field part T/ω

Hard Thermal Loops

Causality and "old" sum rules

$$I_E \equiv \frac{1}{d_A}\int d^3\mathbf{x}\, e^{-i\boldsymbol{q}\cdot\mathbf{x}}\langle E^{i\,a}(t=0,\mathbf{x})E^{i\,a}(0,\mathbf{0})\rangle = \int \frac{d\omega}{2\pi}\,\frac{T}{\omega}\left[2\omega^2\rho_T(\omega,q)+q^2\rho_L(\omega,q)\right] = T\left(2+\frac{m_D^2}{q^2+m_D^2}\right)$$

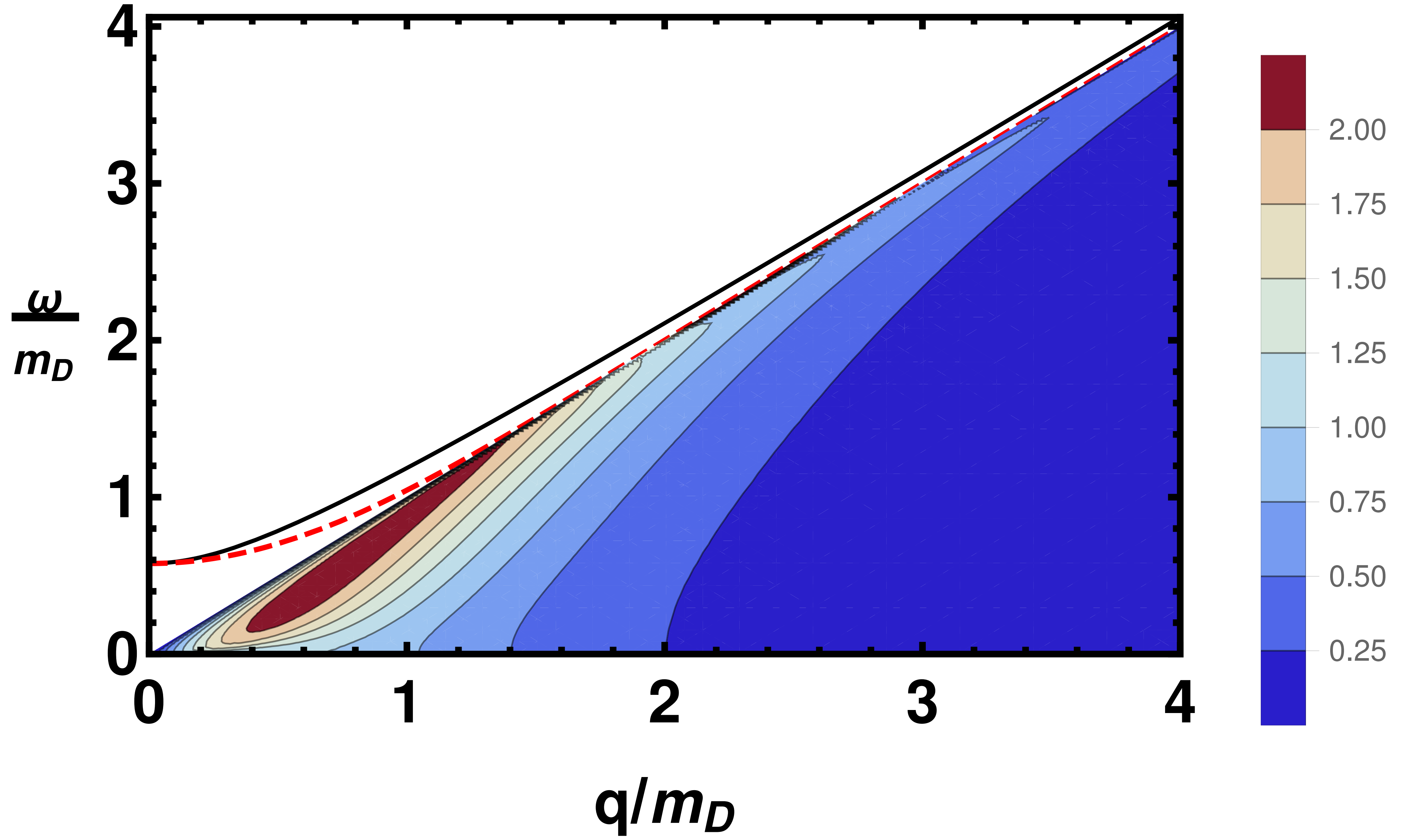
- In the complex plane (see blackboard) the only contributing structures are the zero-mode pole at $\omega = 0$ and the asymptotic behaviour at $|\omega| \to \infty$

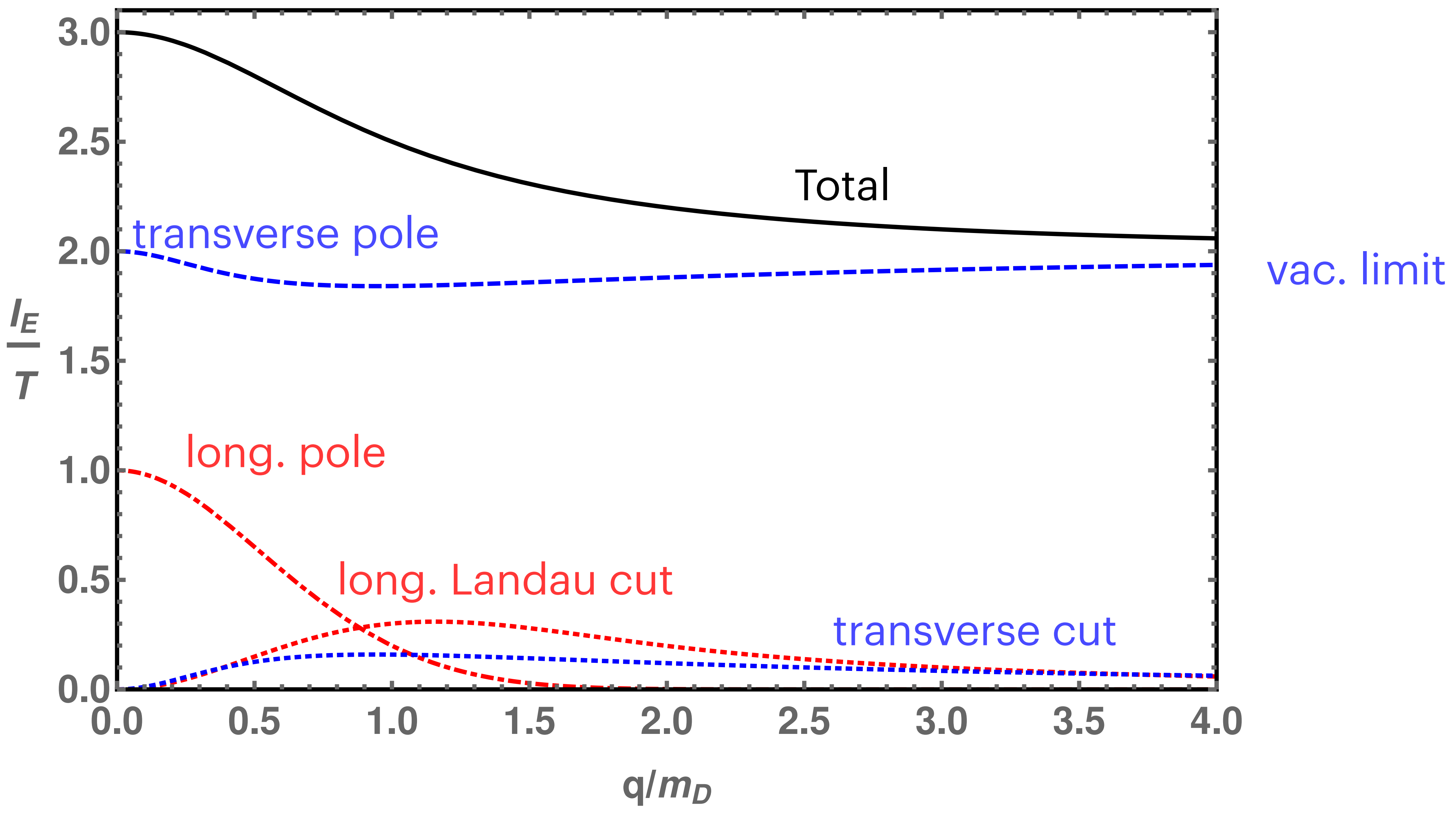
Hard Thermal Loops

Causality and "old" sum rules

$$I_E \equiv \frac{1}{d_A}\int d^3\mathbf{x}\, e^{-i\boldsymbol{q}\cdot\mathbf{x}}\langle E^{i\,a}(t=0,\mathbf{x})E^{i\,a}(0,\mathbf{0})\rangle = \int \frac{d\omega}{2\pi}\frac{T}{\omega}\left[2\omega^2\rho_T(\omega,q)+q^2\rho_L(\omega,q)\right] = T\left(2+\frac{m_D^2}{q^2+m_D^2}\right)$$

- In the complex plane (see blackboard) the only contributing structures are the zero-mode pole at $\omega = 0$ and the asymptotic behaviour at $|\omega| \to \infty$





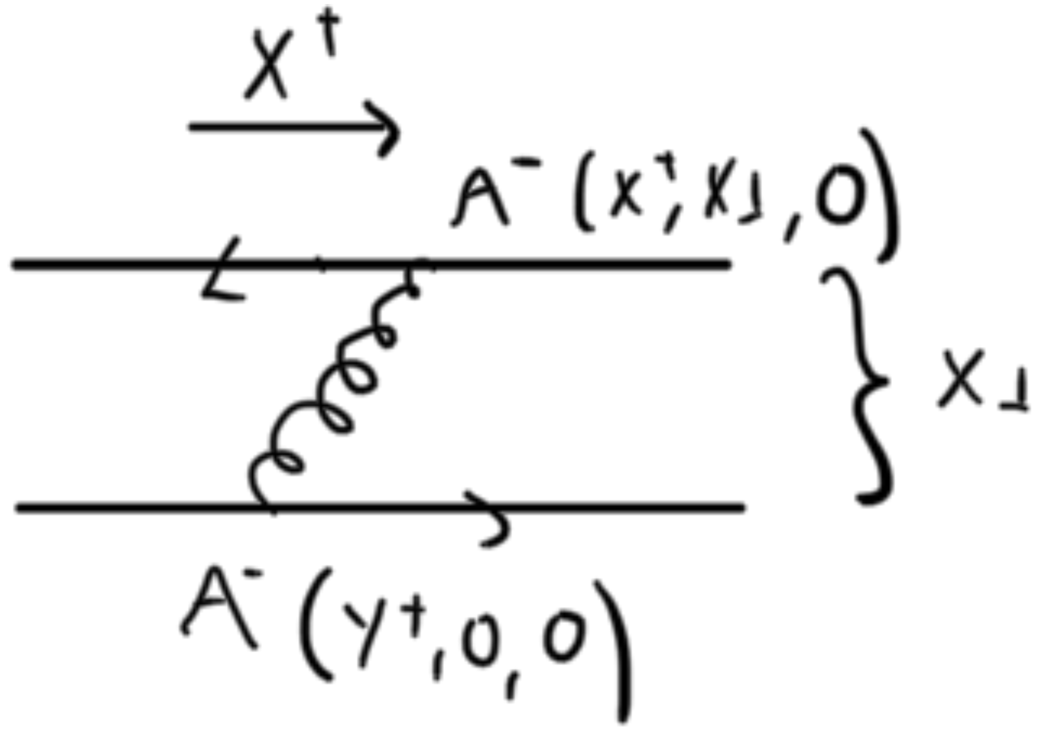
Hard Thermal Loops

Causality and new sum rules

- Let us look at transverse momentum broadening, important ingredient for collinear radiation

$$\mathcal{C}(k_\perp) \equiv (2\pi)^2 \frac{d\Gamma}{d^2k_\perp}, \qquad \hat{q}(\mu) = \int^\mu \frac{d^2k_\perp}{(2\pi)^2} k_\perp^2\, \mathcal{C}(k_\perp)$$

- Hard particle propagating eikonally in the $x^+ = (x^0 + x^z)/2$ direction
- Rate: correlator of eikonal Wilson lines at finite transverse separation Casalderrey-Solana Teaney [hep-th/0701123]



space-like separated

$$C(k_\perp) = g^2 C_R \int dx^+ \int d^2x_\perp\, e^{-ik_\perp \cdot x_\perp} \langle A^-(x^+, x_\perp, 0)\, A^-(0) \rangle$$

Hard Thermal Loops

Causality and new sum rules

- Let us look at transverse momentum broadening

$$\mathcal{C}(k_\perp) = g^2 C_R \int dx^+ \int d^2x_\perp \, e^{-ik_\perp \cdot x_\perp} \langle A^-(x^+, x_\perp, 0) A^-(0) \rangle = g^2 C_R \int \frac{dk^+}{2\pi} G^{--}_{rr}(k^+, k_\perp, 0)$$

$$\hookrightarrow G_{rr} = G_> = G_< \quad \text{as} \quad \vec{x}^2 > x_0^2$$

- Power counting
  - K soft: $g^2 \int dk^+ \frac{T}{k^+} \rho^{--}_{HTL}(K) \sim 1/T$, with $dk^+ \sim gT$, $\frac{T}{k^+} \sim 1/g$, $\rho^{--}_{HTL} \sim 1/g^2T^2$
  - K hard: $g^2 \int dk^+ \left(1/2 + n_B(k^+)\right) \rho^{--}_{1\text{-loop}}(K) \sim g^4/T$, with $dk^+ \sim T$, $\left(1/2 + n_B(k^+)\right) \sim 1$, $\rho^{--}_{1\text{-loop}} \sim g^2/T^2$
- Soft modes dominate total rate, $\int_{k_\perp} \mathcal{C}(k_\perp)$, both LO for $\hat{q} \sim \int_{k_\perp} k_\perp^2 \mathcal{C}(k_\perp)$

Hard Thermal Loops

Causality and new sum rules

- Let us look at transverse momentum broadening

$$C(k_\perp) = g^2 C_R \int dx^+ \int d^2x_\perp\, e^{-ik_\perp \cdot x_\perp} \langle A^-(x^+, x_\perp, 0)\, A^-(0) \rangle \quad = g^2 C_R \int \frac{dk^+}{2\pi}\, G^{--}_{rr}(k^+, k_\perp, 0)$$

$$\hookrightarrow G_{rr} = G_> = G_< \quad \text{at} \quad \vec{x}^2 > x_0^2$$

- This looks complicated for soft gluons
- Space-like separations and causality come to the rescue however, and give us a powerful new(er) “sum rule”: a connection to a dimensionally-reduced Euclidean theory
- Let’s see in detail

Euclideanisation

Causality and analiticity

- For $t/x^z = 0$: equal time correlators: Euclidean (recall first lecture)

$$G_{rr}(t=0,\mathbf{x}) = \sum\!\!\!\!\!\!\int_p G_E(\omega_n,p)e^{i\mathbf{p}\cdot\mathbf{x}}$$

- Consider the more general case $|t/x^z| < 1$

$$G_{rr}(t,\mathbf{x}) = \int dp^0 dp^z d^2p_\perp e^{i(p^z x^z + \mathbf{p}_\perp\cdot\mathbf{x}_\perp - p^0x^0)}\left(\frac{1}{2}+n_{\mathrm{B}}(p^0)\right)(G_R(P)-G_A(P))$$

- Change variables to $\tilde{p}^z = p^z - p^0(t/x^z)$

$$G_{rr}(t,\mathbf{x}) = \int dp^0 d\tilde{p}^z d^2p_\perp e^{i(\tilde{p}^z x^z + \mathbf{p}_\perp\cdot\mathbf{x}_\perp)}\left(\frac{1}{2}+n_{\mathrm{B}}(p^0)\right)(G_R(p^0,\tilde{p}^z+(t/x^z)p^0) - G_A)$$

Euclideanisation

Causality and analiticity

$$D^R(q^+, q^-, \boldsymbol{q}_\perp) = \int dx^+ dx^- d^2x_\perp \; e^{i(q^+x^- + q^-x^+ - \boldsymbol{q}_\perp \cdot \mathbf{x}_\perp)} D^R(x^+, x^-, \mathbf{x}_\perp)$$

- $D^R(X)$ has support for $x^0 > 0$ and $2x^+x^- > x_\perp^2$, i.e. $x^+ > 0$, $x^- > 0$. Hence retarded functions are analytical in the upper plane in any time-like or light-like variable ($q^+ = (q^0 + q^z)/2$, $q^- = q^0 - q^z$)

$$G_{rr}(t, \mathbf{x}) = \int dp^0 d\tilde{p}^z d^2p_\perp e^{i(\tilde{p}^z x^z + \mathbf{p}_\perp \cdot \mathbf{x}_\perp)} \left(\frac{1}{2} + n_{\mathrm{B}}(p^0)\right) (G_R(p^0, \tilde{p}^z + (t/x^z)p^0 - G_A)$$

- G_R analytical in p^0, only poles are the Matsubara modes in n_{B}

$$G_{rr}(t, \mathbf{x}) = T \sum_n \int dp^z d^2p_\perp e^{i(p^z x^z + \mathbf{p}_\perp \cdot \mathbf{x}_\perp)} G_E(\omega_n, p_\perp, p^z {\color{red} + i\omega_n t/x^z})$$

where $\tilde{p}^z$ renamed back to p^z

Euclideanisation

Causality and analiticity

- G_R analytical in p^0, only poles are the Matsubara modes in $n_{\rm B}$

$$G_{rr}(t,\mathbf{x}) = T\sum_n \int dp^z d^2p_\perp e^{i(p^z x^z + \mathbf{p}_\perp\cdot\mathbf{x}_\perp)} G_E(\omega_n, p_\perp, p^z + i\omega_n t/x^z)$$

- Soft physics dominated by $n=0$ (and t-independent) =>EQCD!

$$G_{rr}(t,\mathbf{x})_{\rm soft} = T\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\, G_E(\omega_n = 0, \mathbf{p})$$

- NB: forgot 2π denominators everywhere

Caron-Huot **PRD79** (2009)

Euclideanisation

Causality and analiticity

- Soft physics dominated by $n = 0$ (and t-independent) =>EQCD!

$$G_{rr}(t, \mathbf{x})_{\text{soft}} = T \int d^3p \, e^{i\mathbf{p}\cdot\mathbf{x}} \, G_E(\omega_n = 0, \mathbf{p})$$

- In our case, recalling that $iG_R(\omega, p) = G_E(\omega_E = -i(\omega + i\epsilon), p)$

$$C(k_\perp) = g^2 C_R \int dx^+ \int d^2x_\perp \, e^{-ik_\perp\cdot x_\perp} \langle A^-(x^+, x_\perp) \, A^-(0) \rangle$$

$$= g^2 C_R \int dx^+ \int d^2x_\perp \, e^{-ik_\perp\cdot x_\perp} \, T\int \frac{d^3p}{(2\pi)^3} \, e^{ip_z x^+} \, e^{ip_\perp\cdot x_\perp} \left[G_E^{00}(0,p) + \frac{p_\perp^2}{p^2} G_E^T(0,p) \right]$$

$G_E^{00}(0,p) \to \frac{-1}{p^2 + m_D^2}$, $G_E^T(0,p) \to \frac{1}{p^2}$, $\int dx^+ \to p_z = 0$

$$= g^2 C_R T \left[\frac{1}{k_\perp^2} - \frac{1}{k_\perp^2 + m_D^2} \right]$$

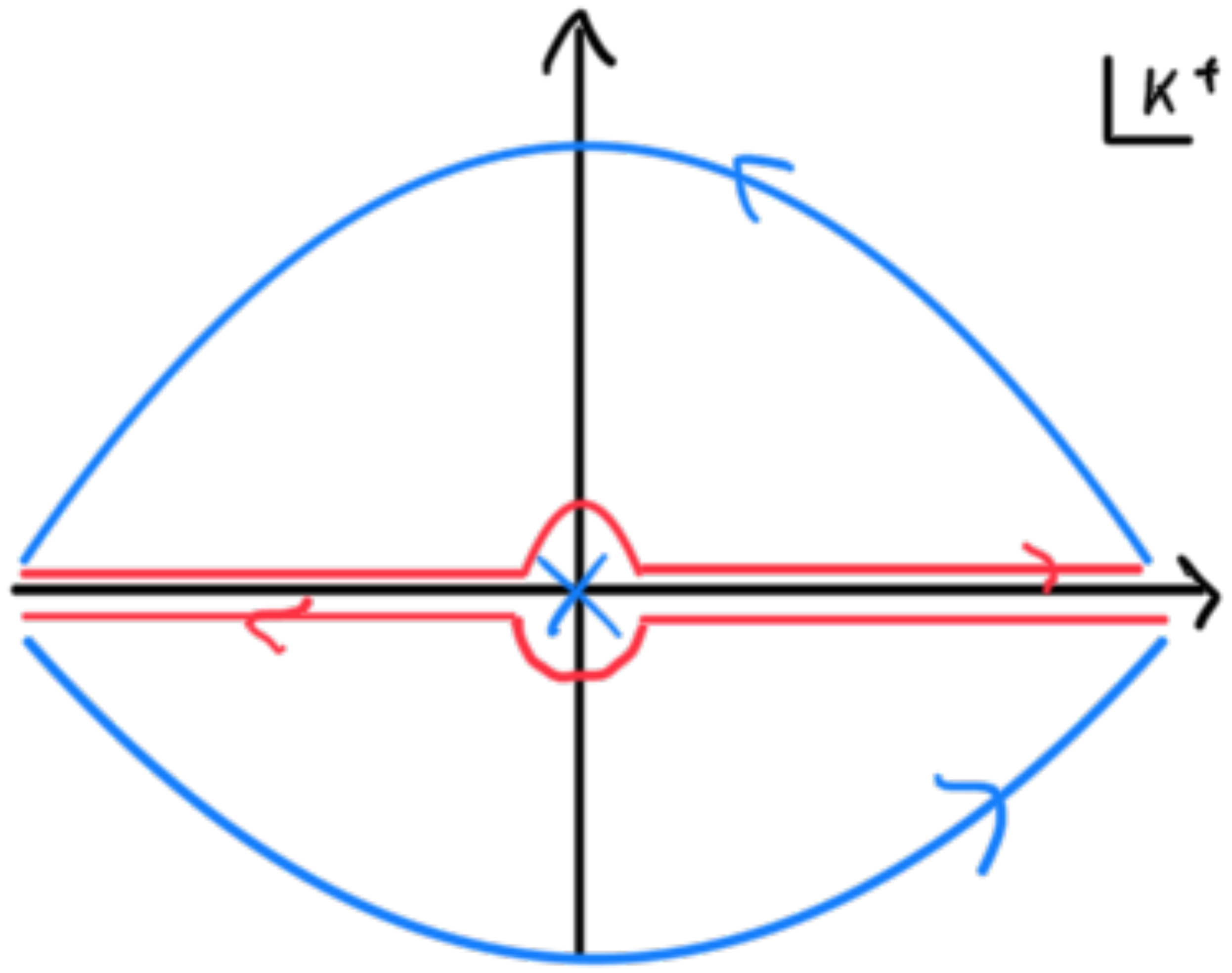
Caron-Huot **PRD79** (2009)

Euclideanisation

Causality and analiticity

- The latest result could have been obtained (for the zero mode only) from the complex plane directly, Aurenche Gelis Zaraket

$$C(k_\perp) = g^2 C_R \int \frac{dk^+}{2\pi} \underbrace{\left(\frac{1}{2} + n_B(k^+)\right)}_{T/k^+} \underbrace{G^{--}}_{G_R^{--} - G_A^{--}}(k^+, k_\perp, 0)$$



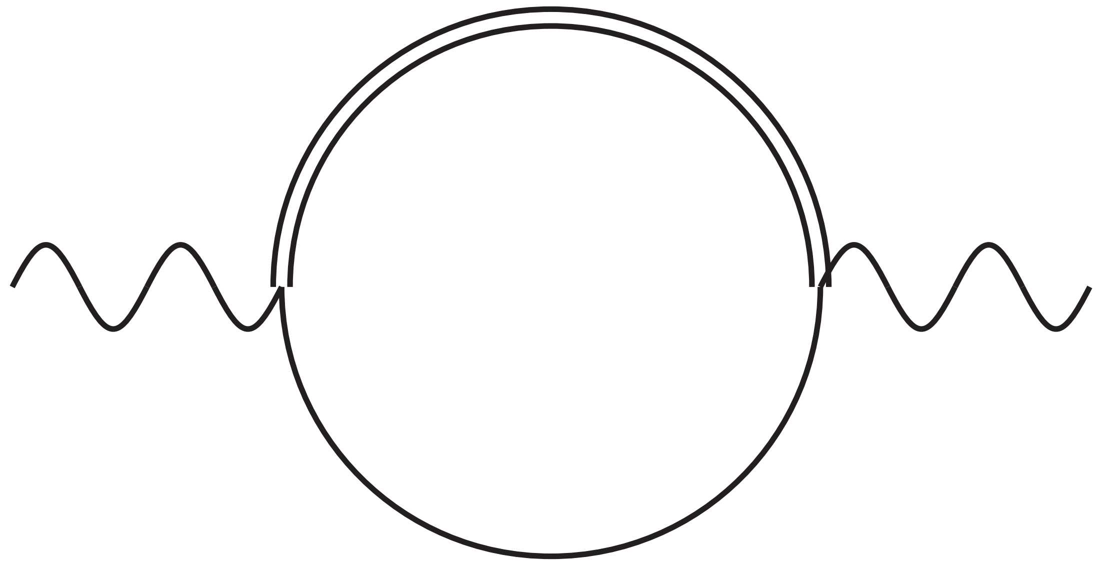
Caron-Huot **PRD79** (2009)

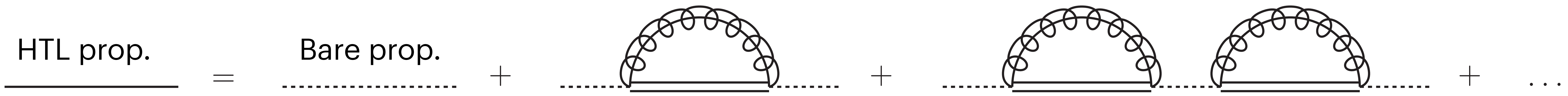
Hard Thermal Loops

Fermionic sum rules and the photon rate

- The proper evaluation of the soft contribution to the photon rate requires HTL resummation for the photon rate. Landau damping will then make this contribution IR finite.

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) =$$





Single line soft, double line hard (they can't be both soft, $k \sim T$). The cut goes through any of the infinite HTLs

Hard Thermal Loops

Fermionic sum rules and the photon rate

- The proper evaluation of the soft contribution to the photon rate requires HTL resummation for the photon rate. Landau damping will then make this contribution IR finite

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = -2\sum_i \int \frac{d^4\mathcal{P}}{(2\pi)^4}\mathrm{Tr}\left[(eQ_i\gamma^\mu)S^{<}_{\mathrm{hard}}(\mathcal{P}+\mathcal{K})(eQ_i\gamma_\mu)S^{<}_{\mathrm{soft}}(-\mathcal{P})\right]$$

- Using the KMS relations this is

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = 2e^2\sum_i Q_i^2 \int \frac{d^4\mathcal{P}}{(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu(-\not{\mathcal{P}}-\not{\mathcal{K}})\gamma_\mu(S_R(\mathcal{P})-S_A(\mathcal{P}))\right]\left(\theta(-p^0-k^0)-n_{\mathrm{F}}(|k^0+p^0|)\right)2\pi\delta((\mathcal{P}+\mathcal{K})^2)(1-n_{\mathrm{F}}(p^0))$$

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = -e^2\sum_i Q_i^2\frac{n_{\mathrm{F}}(k)}{k}\int \frac{d^4\mathcal{P}}{(2\pi)^4}\mathrm{Tr}\left[\not{\mathcal{K}}(S_R(\mathcal{P})-S_A(\mathcal{P}))\right]2\pi\delta(p^-)$$

$\delta(p^-)/2k$ $1/2$

Hard Thermal Loops

Fermionic sum rules and the photon rate

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = -\,e^2 \sum_i Q_i^2 \frac{n_{\mathrm{F}}(k)}{k} \int \frac{d^4\mathcal{P}}{(2\pi)^4} \mathrm{Tr}\big[\not{\mathcal{K}}(S_R(\mathcal{P}) - S_A(\mathcal{P}))\big] 2\pi\delta(p^-)$$

- See where this is going? dp^+ integration, retarded functions
- Recalling $S_R(P) = h_p^+ S_R^+(P) + h_p^- S_R^-(P)$

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = 2e^2 \sum_i Q_i^2 n_{\mathrm{F}}(k) \int \frac{dp^+ d^2p_\perp}{(2\pi)^3} \left[\left(1 - \frac{p^+}{p}\right)\rho^+(p^+, p^- = 0, p_\perp) + \left(1 + \frac{p^+}{p}\right)\rho^-(p^+, p^- = 0, p_\perp)\right]$$

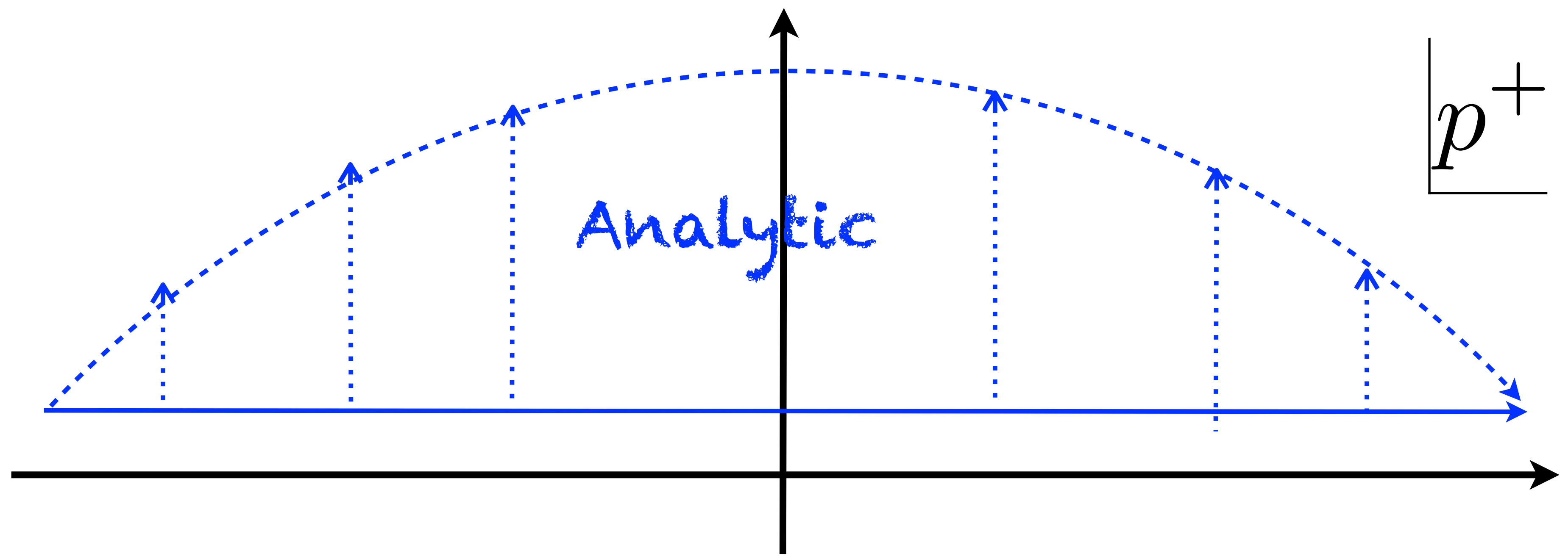
- Clearly, no zero modes $p^+ = 0$ for a fermion.

Hard Thermal Loops

Fermionic sum rules and the photon rate

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = 2e^2 \sum_i Q_i^2 n_{\mathrm{F}}(k) \int \frac{dp^+ d^2p_\perp}{(2\pi)^3} \left[\left(1 - \frac{p^+}{p}\right) \rho^+(p^+, p^- = 0, p_\perp) + \left(1 + \frac{p^+}{p}\right) \rho^-(p^+, p^- = 0, p_\perp) \right]$$

- Deform the dp^+ contour away from the real axis



$$S_R^{\pm}(P) = \frac{i}{p^0 \mp \left[p + \frac{m_\infty^2}{2p} \left(1 - \frac{p^0 \mp p}{2p} \ln\left(\frac{p^0 + p}{p^0 - p} \right) \right) \right]} \Bigg|_{p^0 = p^0 + i\epsilon}$$

$$\left(1 - \frac{p^+}{p}\right) S^+(p^+, p) + \left(1 + \frac{p^+}{p}\right) S^-(p^+, p) \Bigg|_{|p^+| \to +\infty} = \frac{i}{p^+} \frac{m_\infty^2}{p_\perp^2 + m_\infty^2} + \mathcal{O}\left(\frac{1}{(p^+)^2}\right)$$

Hard Thermal Loops

Fermionic sum rules and the photon rate

$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = 2e^2 \sum_i Q_i^2 n_{\mathrm{F}}(k) \int \frac{dp^+ d^2p_\perp}{(2\pi)^3} \left[\left(1 - \frac{p^+}{p}\right) \rho^+(p^+, p^- = 0, p_\perp) + \left(1 + \frac{p^+}{p}\right) \rho^-(p^+, p^- = 0, p_\perp) \right]$$

- Finally one finds

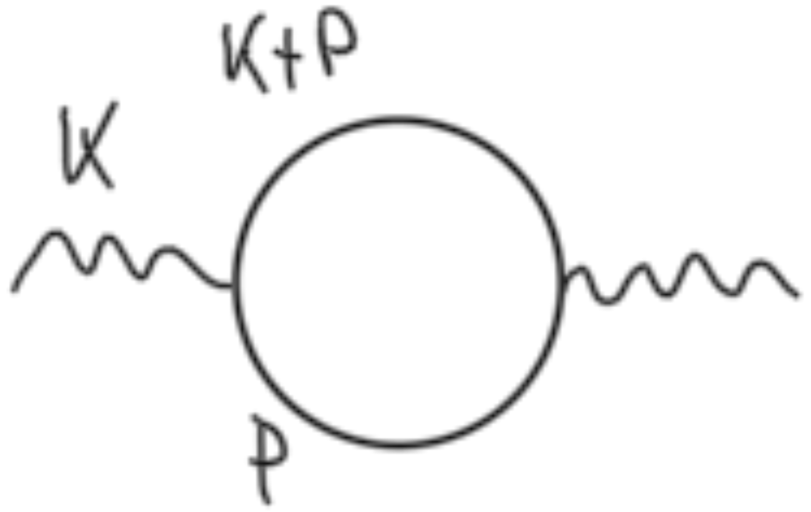
$$\Pi^{<}_{g^2\ \mathrm{soft}}(\mathcal{K}) = 2e^2 \sum_i Q_i^2 n_{\mathrm{F}}(k) \int \frac{d^2p_\perp}{(2\pi)^2} \left[1 - \frac{p_\perp^2}{p_\perp^2 + m_\infty^2} \right]$$

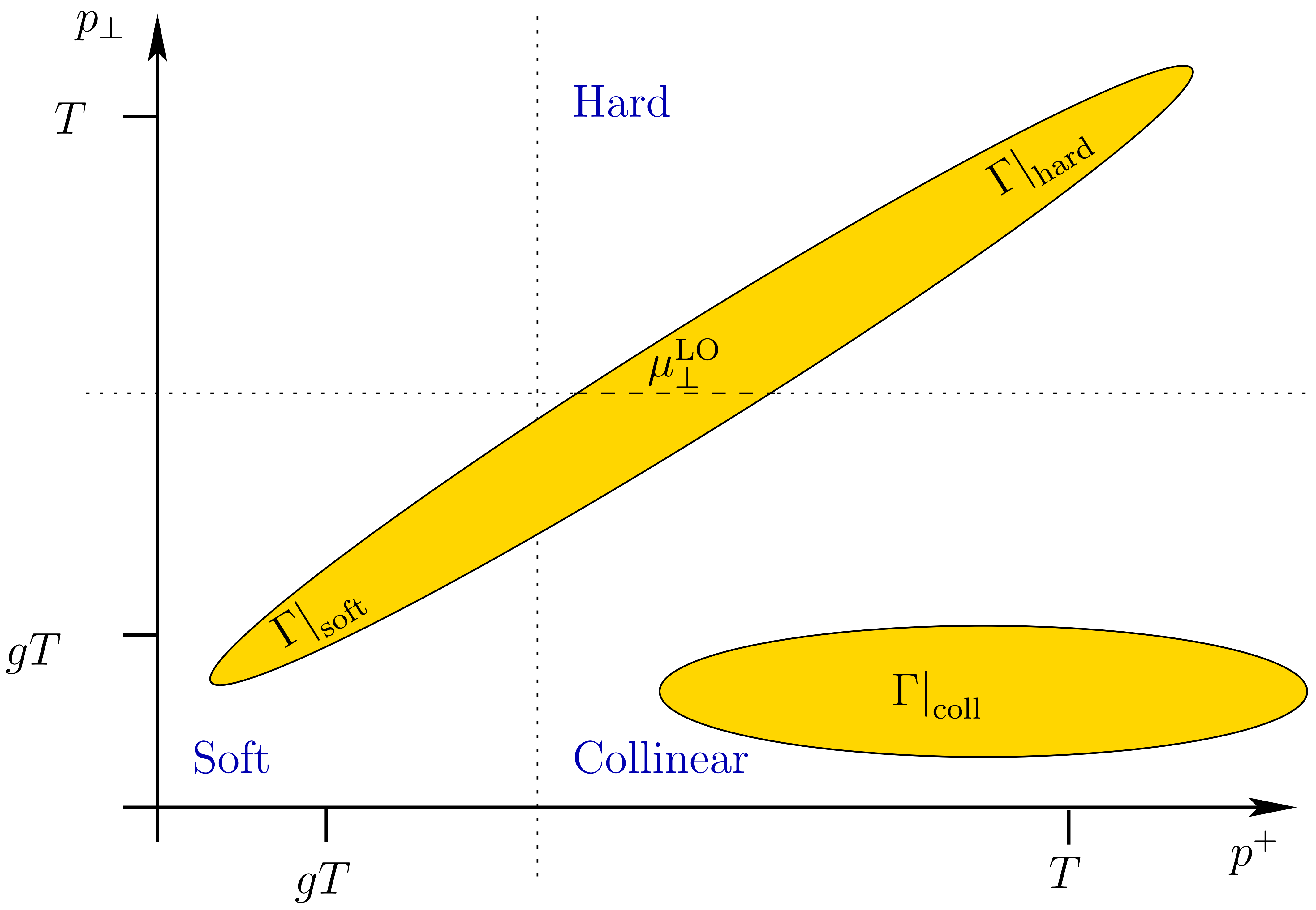
- This is UV log-divergent, removes log sensitivity of naive contribution. A finite result is thus obtained analytically
- Still missing the collinear contribution

Collinear physics

Another failure of the naive loop expansion

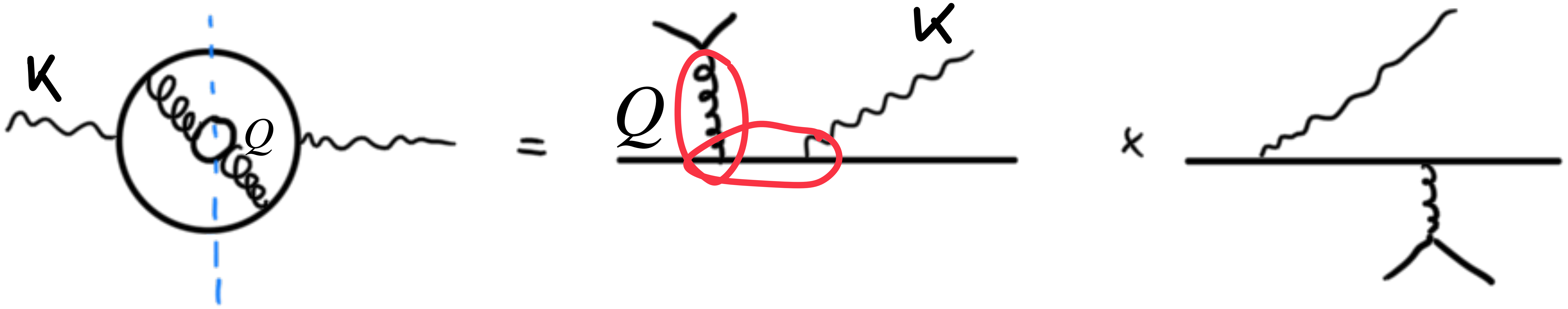
$$\frac{d\Gamma_\gamma}{d^3k} = \frac{\Pi^<(\mathcal{K})}{(2\pi)^3 2k}, \qquad \Pi^<(\mathcal{K}) = \int d^4\mathcal{X} e^{-i\mathcal{K}\cdot\mathcal{X}} \langle J^\mu(0)\, J_\mu(\mathcal{X})\rangle$$





Collinear physics

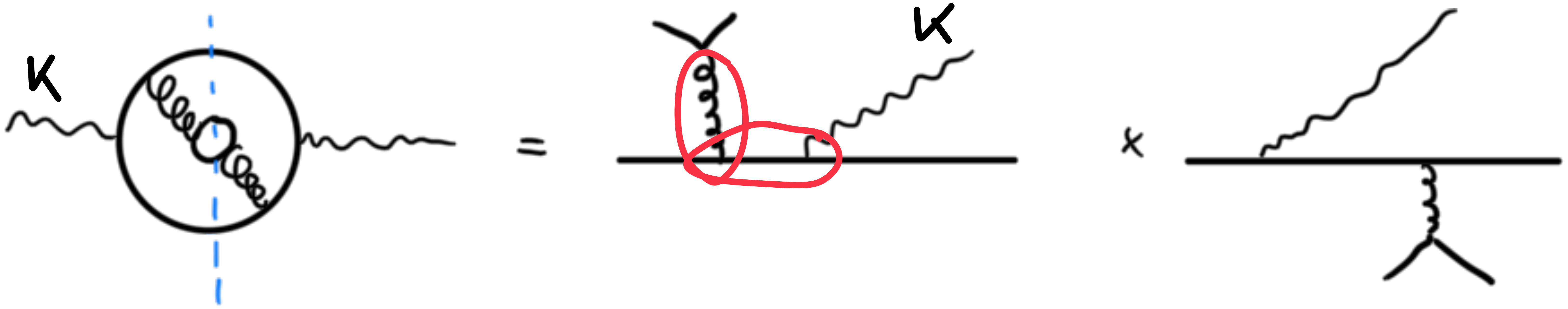
Another failure of the naive loop expansion



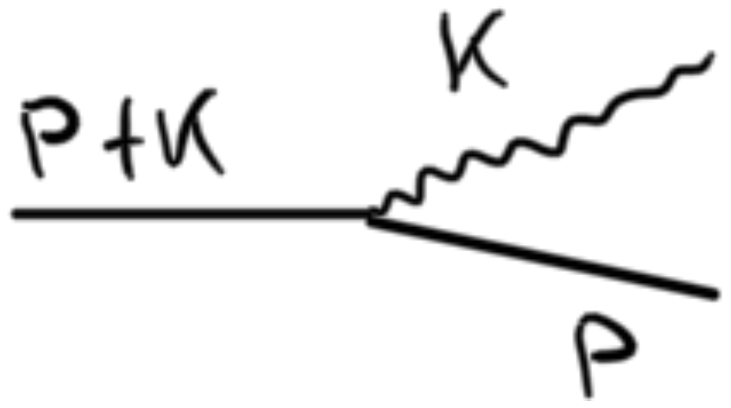
- If $Q \sim gT$ the intermediate quark is close to the mass shell: photon emission is collinear and enhanced
- This Bethe-Heitler process is of the same order as the hard+soft 2<->2 contributions

Collinear physics

Another failure of the naive loop expansion



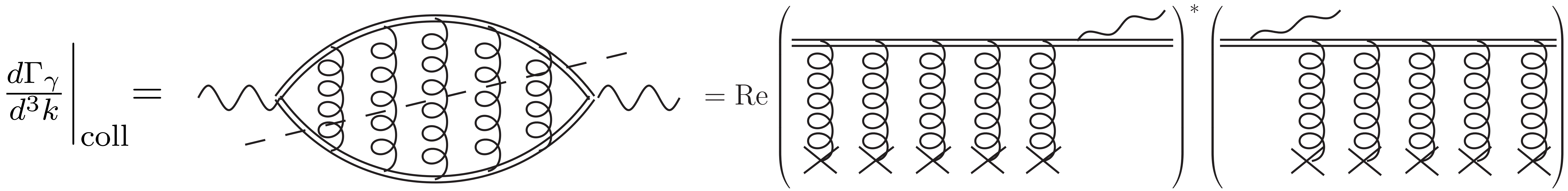
- The formation time is $\dfrac{1}{m_{\text{virt}}}\dfrac{k}{m_{\text{virt}}} = \dfrac{k}{(P+K)^2} = \dfrac{p^+}{p_\perp^2}$



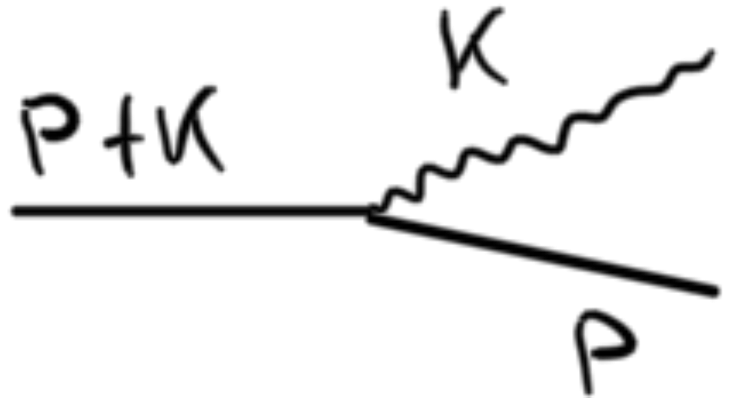
- This is of the same size of the mean free time between soft collisions when $p_\perp \sim gT$

Collinear physics

Another failure of the naive loop expansion



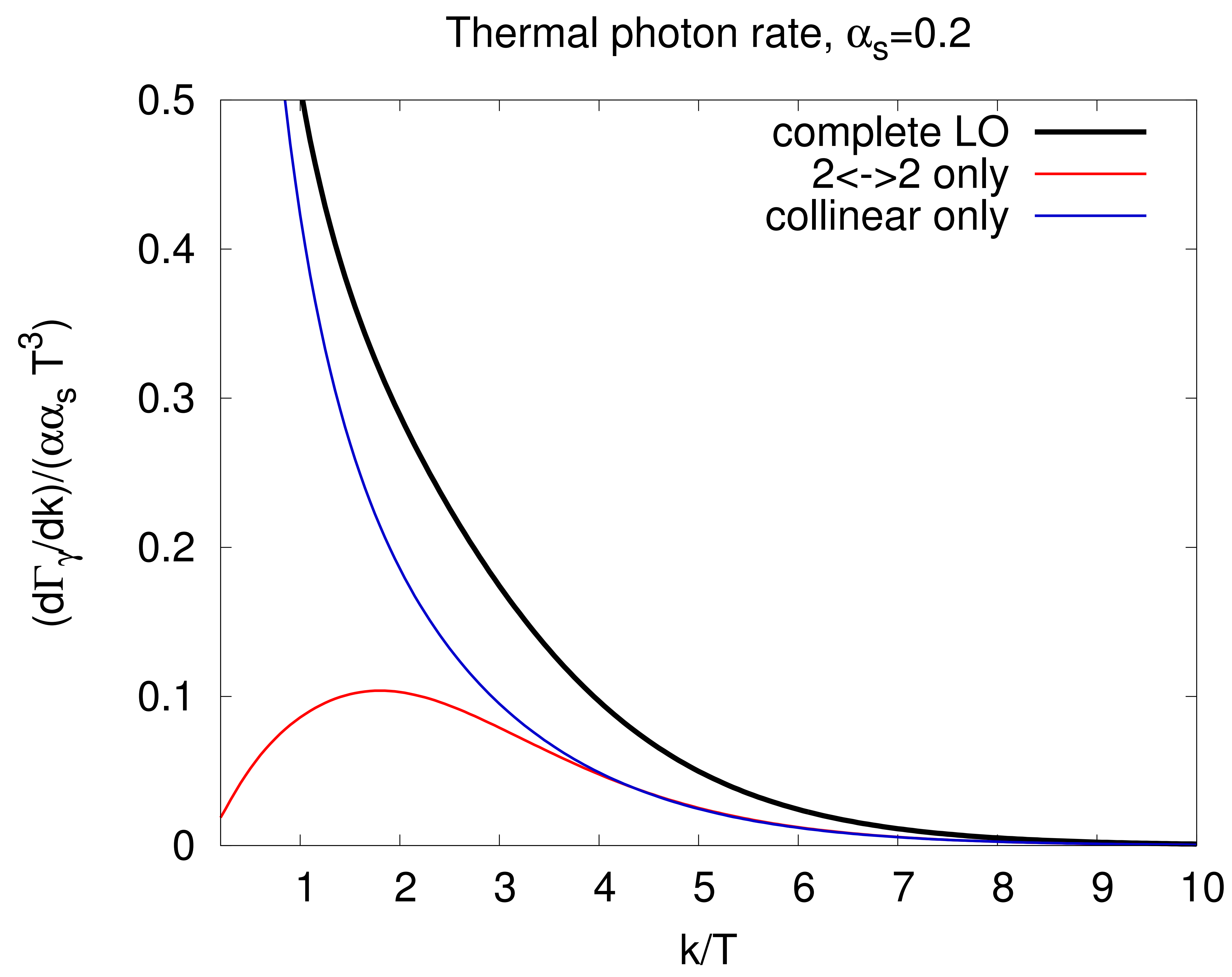
- The formation time is $\dfrac{1}{m_{\rm virt}}\dfrac{k}{m_{\rm virt}} = \dfrac{k}{(P+K)^2} = \dfrac{p^+}{p_\perp^2}$



- This is of the same size of the mean free time between soft collisions when $p_\perp \sim gT$
- Multiple soft interactions have to be resummed, Landau-Pomeranchuk-Migdal effect. Resumming to all orders the $\mathscr{C}$ kernel

The leading-order photon rate

I sat here for 5 hours and all I got was this lousy plot



AMY (Arnold Moore Yaffe) JHEP 0111, 0112, 0226 (2001-02)

Conclusions

- Many modern perturbative calculations of dynamical quantities (not thermodynamics) follow the pattern of the thermal photon rate:
  - multiple scales
  - breakdown of loop expansion
  - HTL resummation for soft modes
  - LPM resummation for collinear modes
- If you are lucky and can map the soft contribution to space-like separated amplitudes, then you can use analyticity to make life much easier