Modern perturbation theory

From Thermal Field Theory to real-time observables

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The plan

- QFT at finite temperature, the role of time and the Schwinger-Keldysh contour
- Bases for real-time perturbation theory
- Soft modes and HTL resummation
- Collinear modes and LPM resummation
- All this in the frame of the LO photon production rate

Bibliography

Never ask the innkeeper if the wine is good

- Perturbative Thermal QCD: Formalism and Applications
  J. Ghiglieri, A. Kurkela, M. Strickland, A.Vuorinen, 2002.10188
  chapters 2-3-4 will be the basis of these lectures
- Basics of Thermal Field Theory
  M. Laine, A.Vuorinen, 1701.01554
  solid reference for perturbative calculations
- Some Aspects of the Theory of Heavy Ion Collisions
  F. Gelis, 2102.07604
  to learn about applications to heavy-ion physics

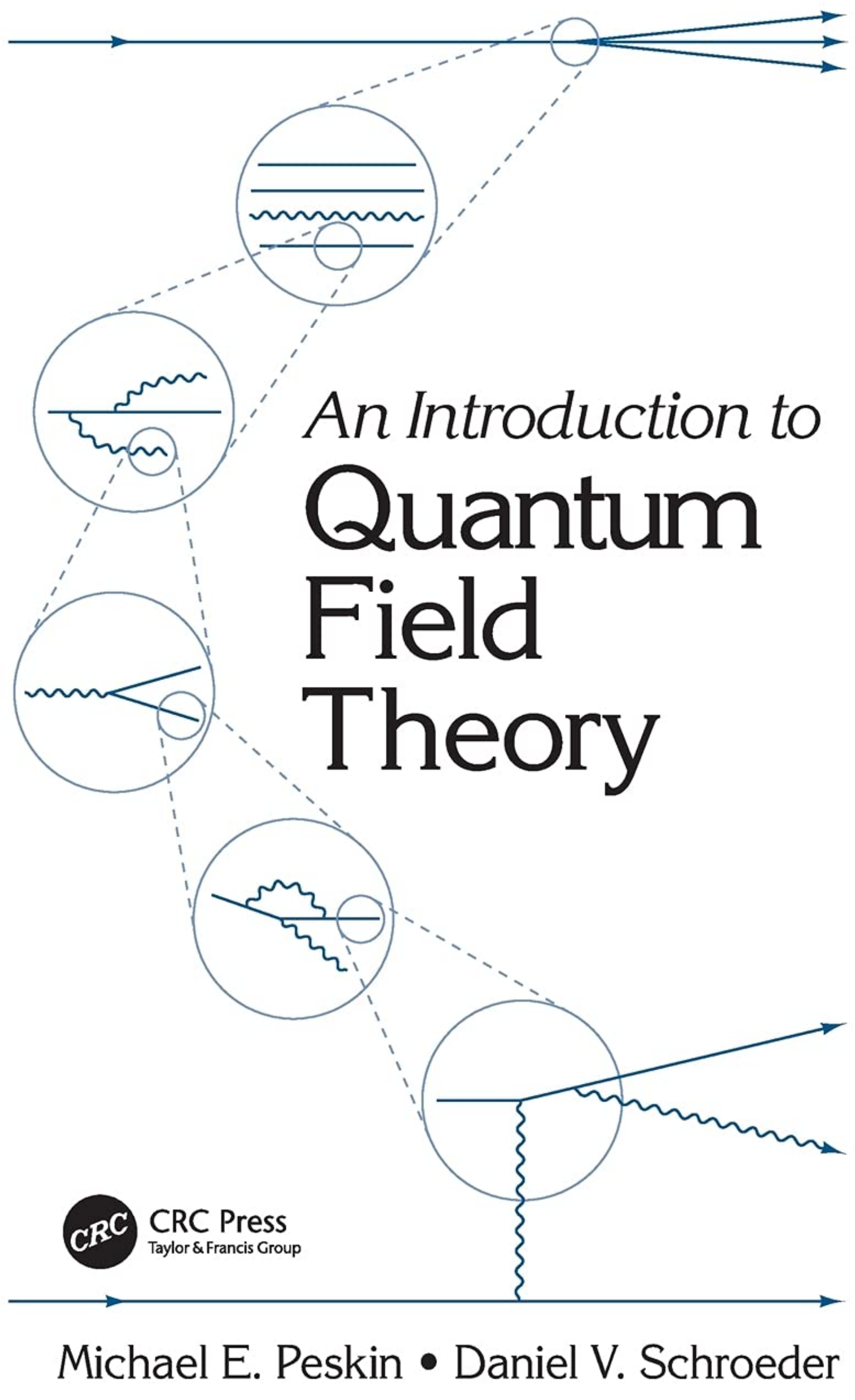
QFT at T=0

A brief recap of what we learned during our master's

- small number of particles
- the primary observable is the scattering amplitude
- we have well defined asymptotic states at $t = \pm\infty$ (or sort of, think of collinear factorisation)

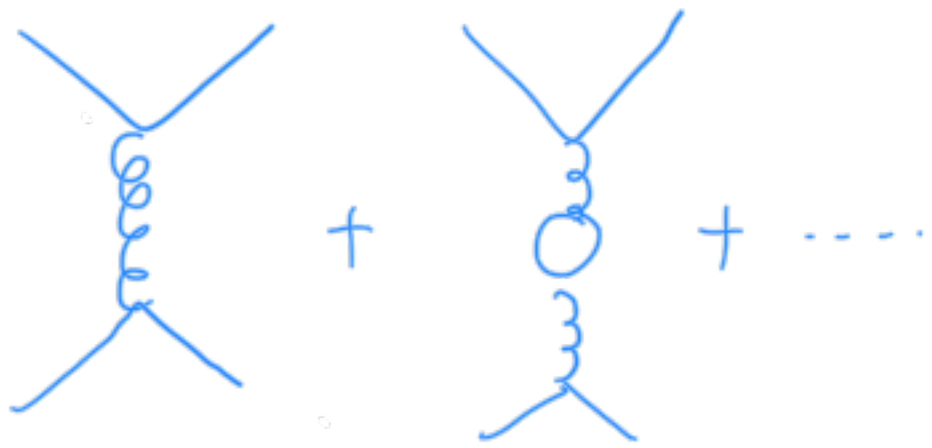
$$\langle \vec{p}_1 \vec{p}_2 | S | \vec{k}_1 \vec{k}_2 \rangle \underset{\text{LSZ reduction}}{\propto} \langle vac.| T\{\varphi(x_1)\varphi(x_2)\varphi(y_1)\varphi(y_2)\} | vac.\rangle$$

vacuum to vacuum amplitudes, fields act on vacuum to create the asymptotic states



QFT at T=0

A brief recap of what we learned during our master's

- Perturbatively 

  includes loops: quantum-mechanical **vacuum** fluctuations (Heisenberg principle)

- Indeed, the vacuum is a quantum-mechanical **pure state**. Vacuum expectation values only accounts for vacuum fluctuations

QFT in a medium
Statistical fluctuations

- In a medium we also have statistical fluctuations, arising from our *limited, statistical knowledge* of the system
- If at $t = t_0$ the system is described by states $|i\rangle$ with **probabilities** $P_i(t_0)$ (a mixed state) then

$$\langle \hat{O}(t_0) \rangle \equiv \sum_i P_i(t_0) \langle i | \hat{O} | i \rangle = \mathrm{Tr}[\hat{\rho}(t_0) \hat{O}(t_0)]$$

with $\hat{\rho}(t_0) = \sum_i P_i(t_0) |i\rangle\langle i|$ the density operator

- In what follows we will concentrate on **thermal equilibrium.** Out-of-equilibrium is very fascinating&important, see Aleksi's lectures

Thermal equilibrium

The grand canonical ensemble

- The density operator is now time-independent

$$\hat{\rho}_{\text{eq}} = \frac{1}{Z} e^{-\beta(\hat{H} - \mu_i \hat{N}_i)}, \qquad Z = \text{Tr}\, e^{-\beta(\hat{H} - \mu_i \hat{N}_i)},$$

- Z is the partition function (Zustandssumme), $\beta \equiv 1/T$, $\hat{H}$ the Hamiltonian, $\hat{N}_i$ the number operators for conserved global charges, with associated chemical potentials μ_i
- For instance, in QCD $\quad \hat{N}_f = \int d^3x\, \bar{q}_f(x) \gamma^0 q_f(x)$

Observables

- Typical observables
  - thermodynamics (p, e, susceptibilities…)
  - transport coefficients (η, diffusion,…)
  - thermal production rates
  - hard-probe observables (jets, quarkonia)
  - equilibration and thermalisation rates
  - ……

Classifying observables

An important difference

- Even if the equilibrium state is time-independent, we can classify these observables by how they are affected by time
- For thermodynamics, $T^{\mu\nu} = \mathrm{diag}(e, p, p, p)$ for an ideal fluid in its rest frame. In QFT $T^{\mu\nu} \to \Theta^{\mu\nu}$, which is a local operator ($\Theta^{\mu\nu}(X)$). Then $e = \langle \Theta^{00} \rangle$ (with vacuum subtraction)
- Thermodynamics deals with operators which are local in time. As we shall soon see, that is a big simplification
- These lectures will mostly be about observables that are non-local in time

Dealing with local observables

The Matsubara formalism

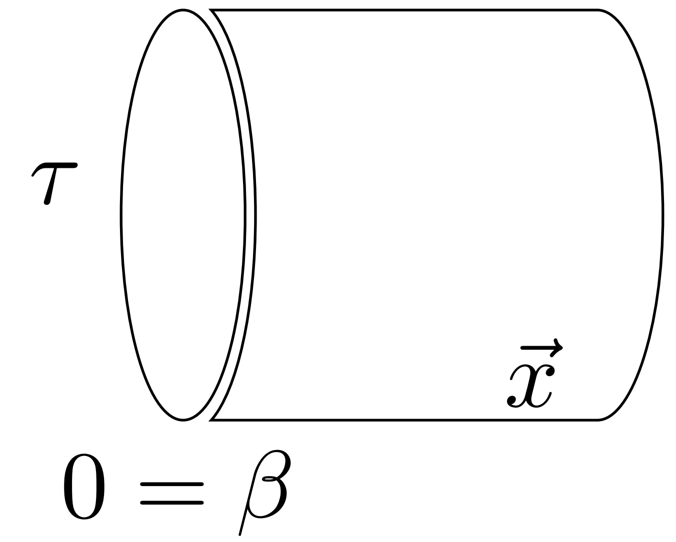
- $\langle \hat{O}(t) \rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{O}(t)] = \sum_i \langle i \,|\, \hat{\rho}(t)\hat{O}(t) \,|\, i \rangle$: used *t*-invariance of eq. operator
- Now use $\hat{\rho} = e^{-\beta(\hat{H}-\mu_i \hat{N}_i)}/Z$ and identify $e^{-\beta \hat{H}}$ as a time-evolution operator in the imaginary direction τ, i.e $it \leftrightarrow \beta$ ($U(t) = e^{-i\hat{H}t}$)

$$\langle \hat{O} \rangle = \frac{\int \mathcal{D}\phi \, \hat{O} \, e^{-S_E}}{\int \mathcal{D}\phi \, e^{-S_E}} \qquad S_E \equiv \int_0^\beta d\tau L_E$$

- The trace, when transformed into a path integral, implies
  $\phi(0, \overrightarrow{x}) = \phi(\beta, \overrightarrow{x})$ for bosons (periodicity)
  $\psi(0, \overrightarrow{x}) = -\psi(\beta, \overrightarrow{x})$ for fermions (antiperiodicity)

Dealing with local observables

The Matsubara formalism



- We thus have 3D Euclidean space X compactified Euclidean time
- Ideal (at vanishing chem. pots) for lattice
- Perturbatively: Euclidean field theory with discrete *Matsubara frequencies* $\omega_n = 2\pi T n$ for bosons, $\tilde{\omega}_n = \pi T(2n+1)$ for fermions, $n \in \mathbb{Z}$.
  $\int d\omega/(2\pi) \to T \sum_n$
- **Exercise**: for a theory of massless, non-interacting real scalars compute the energy density using dim reg. Recall that $\Theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{\delta_{\mu\nu}}{2} \partial_\rho \phi \partial_\rho \phi$.

  Solution: $e = 4\pi^2 T^4 \zeta(-3) = \dfrac{\pi^2 T^4}{30}$

The thermal photon rate
A non-local observable

- Photons are not in equilibrium in the short-lived QCD plasma in a heavy-ion collision
- If a photon is produced from the QCD plasma (rare event, $\alpha \ll 1$) it is unlikely to rescatter (rare event, $\alpha \ll 1$)^2
- They are thus a good **hard probe**: they carry information from the thermal phase, unaffected by later stages such as hadronisation

The thermal photon rate

A non-local observable

- Since $\alpha \ll 1$ we can work to first order in the EM coupling. Photon production is then Poissonian and back-reaction (cooling of the plasma) negligible
- Compute single photon production

$$2k(2\pi)^3 \frac{dP}{d^3k} = \sum_X \mathrm{Tr}\rho_{\mathrm{eq}} U^\dagger(t)|X,\gamma\rangle\langle X,\gamma|U(t)$$

P probability, *U(t)* time evolution operator

- The rate is

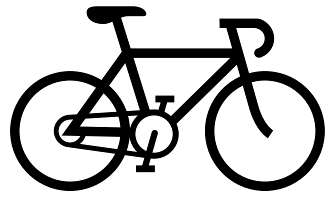
$$\frac{dN_\gamma}{d^4X d^3k} \equiv \frac{d\Gamma_\gamma}{d^3k} \overset{k\parallel z}{=} \frac{-e^2}{(2\pi)^3 2k} \int d^4X e^{ik(t-z)} \langle J^\mu(0) J_\mu(X)\rangle$$

The thermal photon rate
A non-local observable

- The rate is $$\frac{dN_\gamma}{d^4X d^3k} \equiv \frac{d\Gamma_\gamma}{d^3k} \overset{k\|z}{=} \frac{-e^2}{(2\pi)^3 2k} \int d^4X e^{ik(t-z)} \langle J^\mu(0) J_\mu(X) \rangle$$
- Two-point function of the em current $J^\mu = \sum_i Q_i \bar{q}_i \gamma^\mu q_o$
- $\langle J^\mu \rangle$ would be the current density of the plasma. Vanishes for a charge-neutral plasma. But a charge-neutral plasma still produces photons
- The formula is valid to first order in electromagnetism but to all orders in the QCD coupling. For practical and pheno reasons require $k \sim T$

Dealing with non-local observables

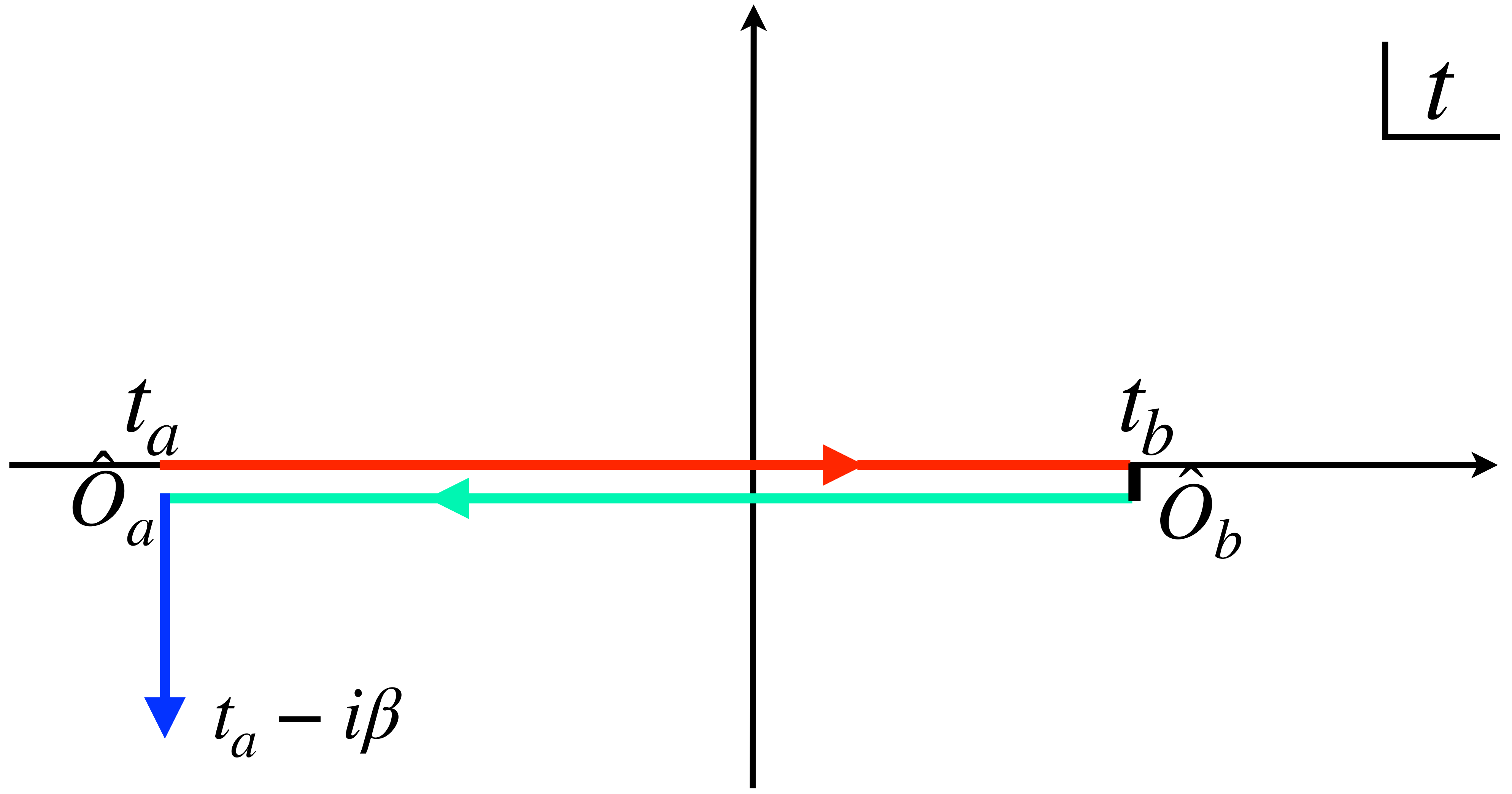
The Schwinger-Keldysh formalism

- $\langle \hat{O}_a(t_a)\hat{O}_b(t_b)\rangle = \mathrm{Tr}[\hat{\rho}(t_a)\hat{O}_a(t_a)\hat{O}_b(t_b)]$: chose t_a for the density op. and assume $t_b > t_a$
- $\hat{\rho}_a(t_a) \equiv \hat{\rho}(t_a)\hat{O}_a(t_a)$ can be considered as a new, non-equilibrium state
- $[\hat{H}, \hat{\rho}_a] \neq 0$, we must evolve it to t_b, $\hat{\rho}_a(t_b) = U(t_b, t_a)\hat{\rho}_a(t_a)U(t_a, t_b)$
- Plug this into the trace above and use **cyclicity** 
  $\mathrm{Tr}[\hat{\rho}_a(t_b)\hat{O}_b(t_b)] = \mathrm{Tr}[\hat{\rho}(t_a)\hat{O}_a(t_a)U(t_a, t_b)\hat{O}_b(t_b)U(t_b, t_a)]$
- Translating this into path integrals we get the **Schwinger-Keldysh contour**

The Schwinger-Keldysh contour

$$\mathrm{Tr}[\hat{\rho}_a(t_b)\hat{O}_b(t_b)] = \mathrm{Tr}[\hat{\rho}(t_a)\hat{O}_a(t_a)U(t_a,t_b)\hat{O}_b(t_b)U(t_b,t_a)]$$

- Time ordered branch
- Anti-time ordered branch
- Statistical (Euclidean) branch



$$O_{li} = \langle l|\hat{O}_a\hat{O}_b|i\rangle$$

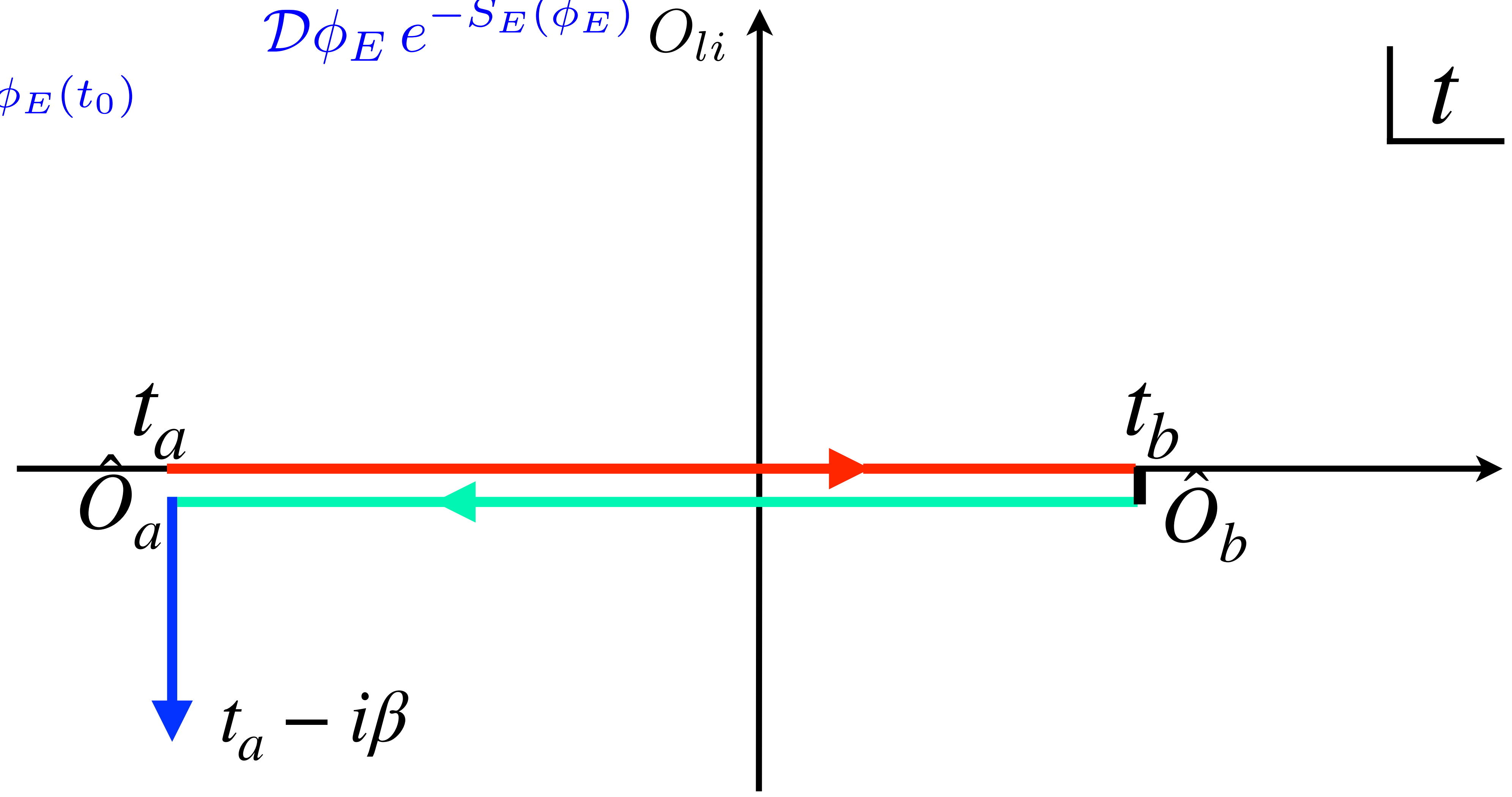
$$\langle \hat{O}_a(t_a)\hat{O}_b(t_b)\rangle = \frac{1}{Z}\sum_{li}\int_{\phi_1(t_a)=\phi_E(t_0-i\beta)}^{\phi_1(t_b)=\phi_i}\mathcal{D}\phi_1\int_{\phi_2(t_a)=\phi_E(t_0)}^{\phi_2(t_b)=\phi_l}\mathcal{D}\phi_2\, e^{i(S(\phi_1)-S(\phi_2))}\int_{\phi_E(t_0)}^{\phi_E(t_0-i\beta)}\mathcal{D}\phi_E\, e^{-S_E(\phi_E)}\, O_{li}$$

- Applicable out of equilibrium too
- ~~Doubling of the degrees of freedom~~, an imprecise statement

The Schwinger-Keldysh contour

~~Doubling of the degrees of freedom~~

$$\langle \hat{O}_a(t_a)\hat{O}_b(t_b)\rangle = \frac{1}{Z}\sum_{li}\int_{\phi_1(t_a)=\phi_E(t_0-i\beta)}^{\phi_1(t_b)=\phi_i}\mathcal{D}\phi_1\int_{\phi_2(t_a)=\phi_E(t_0)}^{\phi_2(t_b)=\phi_l}\mathcal{D}\phi_2\, e^{i(S(\phi_1)-S(\phi_2))}\int_{\phi_E(t_0)}^{\phi_E(t_0-i\beta)}\mathcal{D}\phi_E\, e^{-S_E(\phi_E)}\, O_{li}$$



- An imprecise statement: as we shall see, the propagator becomes a matrix in 1 and 2 fields. If one computes a time-ordered product of operators, then external fields are of type 1 and internally these ``2'' dofs pop in. But T-ordered operators are not that important in thermal eq. (with some exceptions

- The more important operator orderings are the Wightman $\langle \phi(t)\phi(0)\rangle$ and retarded two-point fncs. They measure correlation and causation respectively

The Schwinger-Keldysh contour
Operator orderings

- Wightman functions $D^>(t_1, t_0) = \langle \phi(t_1)\phi(t_0) \rangle$ $D^<(t_1, t_0) = \langle \phi(t_0)\phi(t_1) \rangle$
- Retarded and advanced functions $D^R(t_1, t_0) = \theta(t_1 - t_0)\rho_B(t_1, t_0),$ $D^A(t_1, t_0) = -\theta(t_0 - t_1)\rho_B(t_1, t_0)$
- Their difference: the *spectral function* $\rho_B(t_1, t_0) = \langle [\phi(t_1), \phi(t_0)] \rangle$

  $\rho_B(t_1, t_0) = D^R(t_1, t_0) - D^A(t_1, t_0) = D^>(t_1, t_0) - D^<(t_1, t_0)$

- Cyclicity of the trace and the (exponential) form of eq. density yield the **Kubo-Martin-Schwinger (KMS)** relation between the Wightman functions $D^>(t) \equiv D^>(t,0) = D^<(t + i\beta)$, $S^>(t) = - e^{\beta\mu} S^<(t + i\beta)$

The Schwinger-Keldysh contour

Operator orderings

- In Fourier space

$$D^>(\omega) \equiv \int dt e^{i\omega t} D^>(t) = e^{\beta\omega} D^<(\omega),$$

$$S^>(\omega) \equiv \int dt e^{i\omega t} S^>(t) = -e^{\beta(\omega-\mu)} S^<(\omega)$$

- Hence we get the Wightman functions in terms of the spectral function

$$D^<(\omega) = (e^{\beta\omega} - 1)^{-1}\rho_B(\omega) \equiv n_B(\omega)\rho_B(\omega),$$

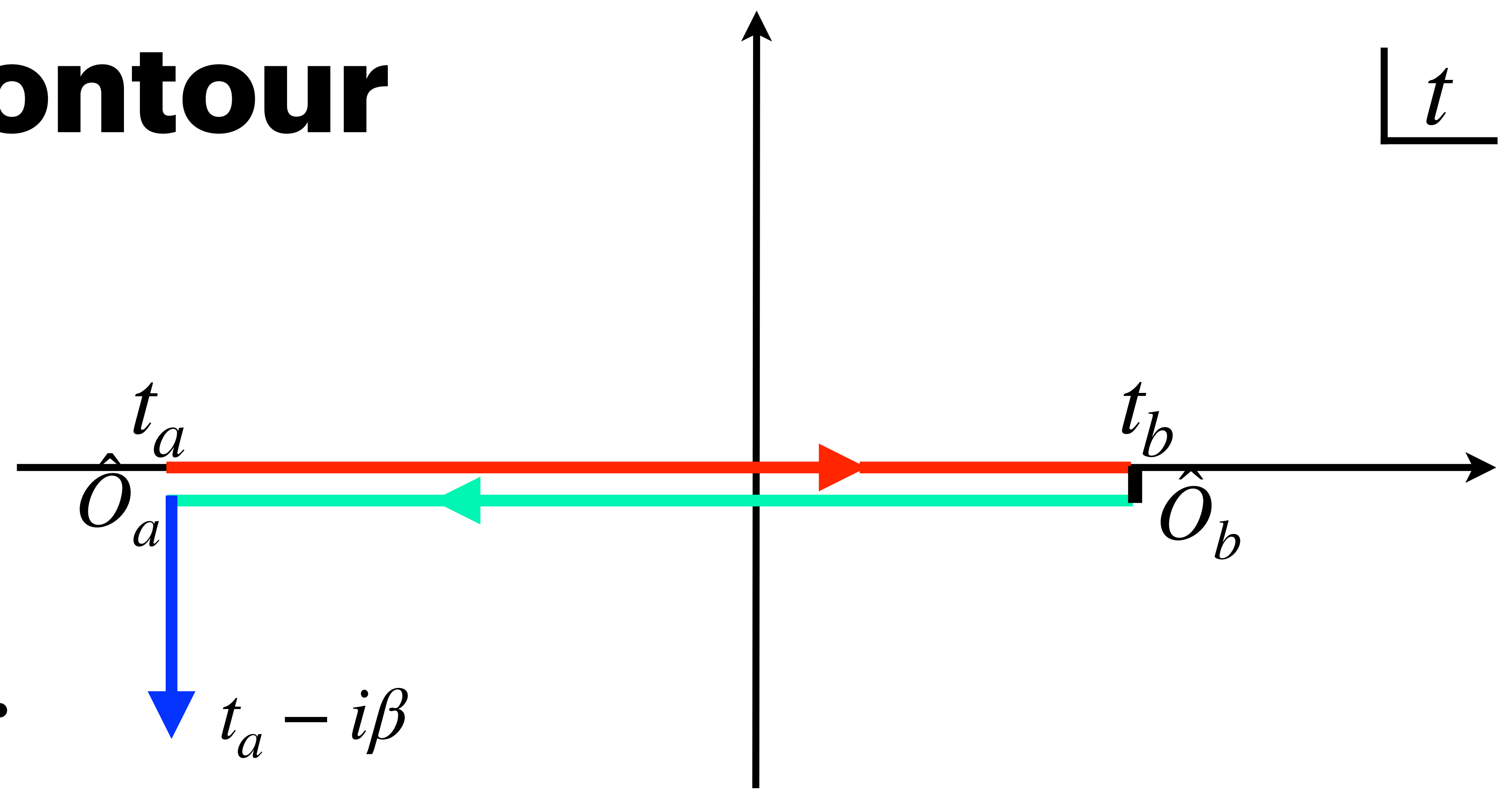
$$D^>(\omega) = (1 + n_B(\omega))\rho_B(\omega)$$

$$S^<(\omega) = -(e^{\beta(\omega-\mu)} + 1)\rho_F(\omega) \equiv -n_F(\omega)\rho_F(\omega)$$

- In equilibrium **all operator orderings** are determined from the spectral function through KMS and causality

The ``1-2'' formalism

Operator orderings and the contour



- Go back to the contour: ``2'' fields are always to the left (contour-later) of ``1'' fields,
  $\langle \phi_2(t)\phi_1(0)\rangle = D^{>}(t), \qquad \langle \phi_1(t)\phi_2(0)\rangle = D^{<}(t).$
  This determines the off-diagonal elements of the propagator.

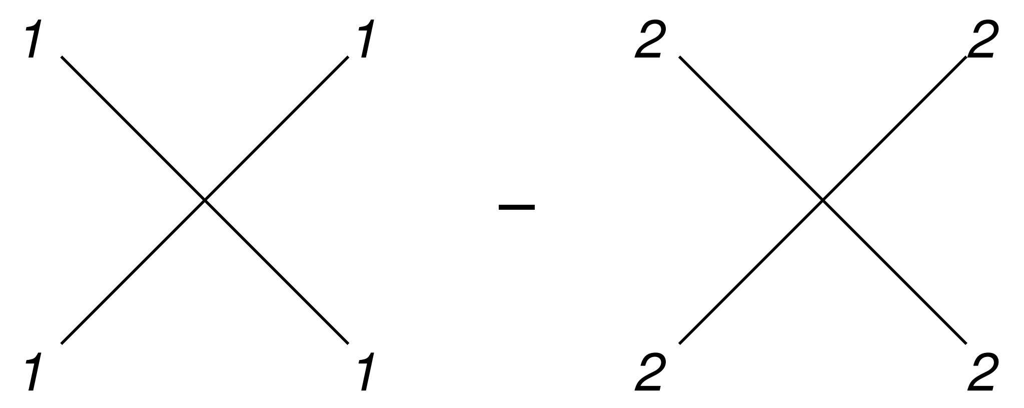
- The two contours are (anti)-time ordered ⇒ Feynman and anti-F propagators

$$\mathbf{D} = \begin{pmatrix} \langle \phi_1\phi_1\rangle & \langle \phi_1\phi_2\rangle \\ \langle \phi_2\phi_1\rangle & \langle \phi_2\phi_2\rangle \end{pmatrix} = \begin{pmatrix} D^F & D^< \\ D^> & D^{\bar{F}} \end{pmatrix}$$

$$D^{F,\bar{F}}(t) = \theta(\pm t)D^>(t) + \theta(\mp t)D^<(t)$$

$$D^{F,\bar{F}}(\omega) = \pm\frac{1}{2}[D_R(\omega) + D_A(\omega)] + \left(\frac{1}{2} + n_B(\omega)\right)\rho_B(\omega)$$

- Vertices are diagonal in ``1-2'' indices, ``2'' indices have opposite sign



The ``1-2'' formalism

Operator orderings and the contour

- We can now go back to our energy density problem and see how it appears more transparent in the real-time formalism

$$e = \int \frac{d^4P}{(2\pi)^4} p^2 D^>(P)$$

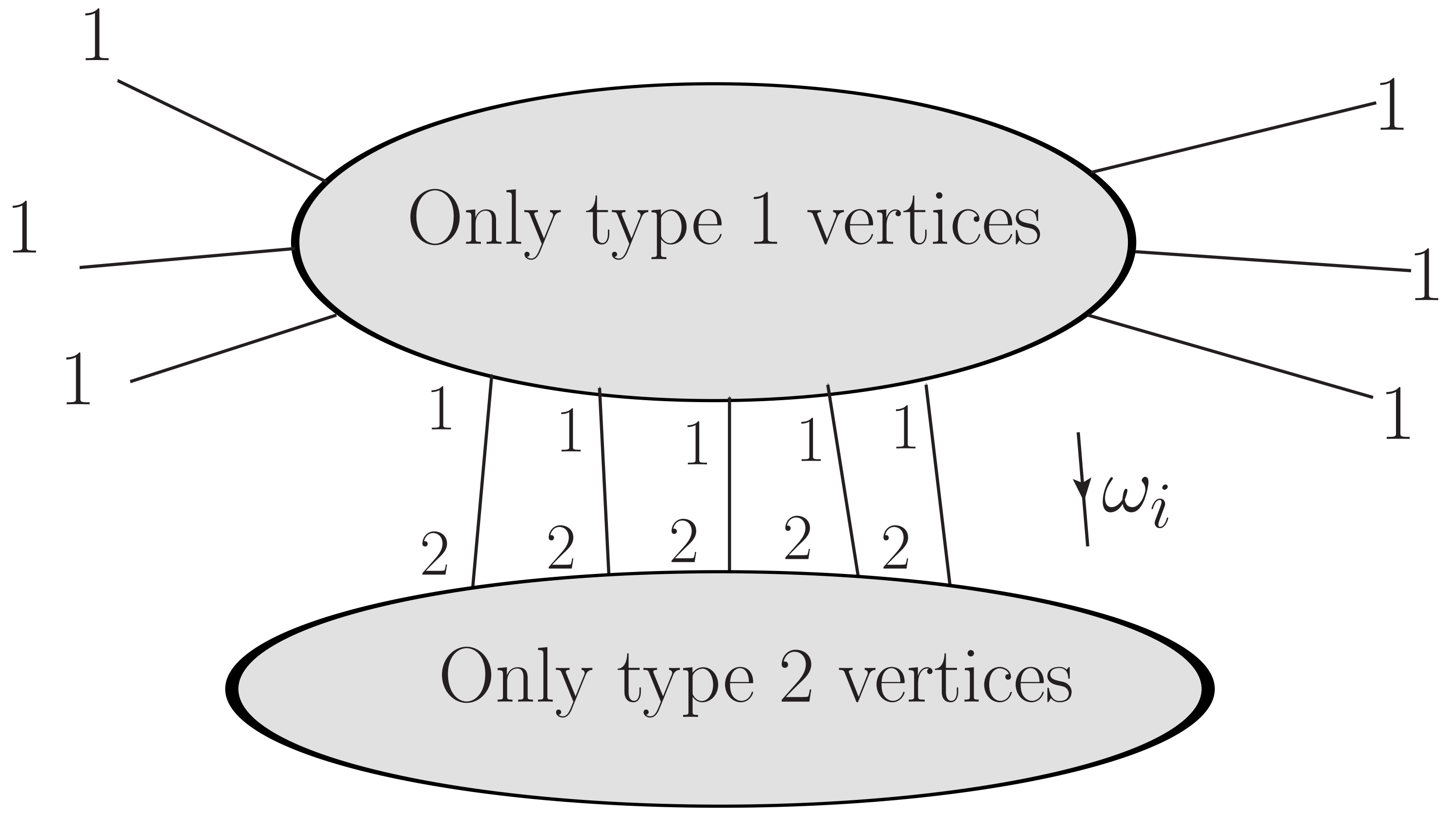
- Recall that the bare spf is $\rho_B(\omega) = 2\pi\epsilon(\omega)\delta(\omega^2 - E_k^2)$
- Important: time-dependent observables can also be obtained from the Matsubara formalism, but one needs to perform analytical continuations (after all the sums have been performed).

$$D^>(-i\tau) = G_E(\tau)$$

The ``1-2'' formalism

Why there is no ~~doubling~~ in vacuum

- For a time-ordered vacuum expectation value all external field are type ``1''. ``2'' fields can only appear in islands



- They are thus connected to the rest of the diagram by $D^{>}(\omega_i)$ propagators

- $n \geq 1$ ω_i have to be negative by momentum conservation.

- Recall that $D^{>}(\omega) = (1 + n_B(\omega))\rho_B(\omega) \to \theta(\omega)\rho_B(\omega)$ at zero temperature: the familiar statement that forward Wightman vac. amplitudes have support at positive frequencies only is what prevents these islands from popping up

The ``1-2'' formalism

Are we ready to go?

$$\frac{dN_\gamma}{d^4X d^3k} \equiv \frac{d\Gamma_\gamma}{d^3k} \stackrel{k\parallel z}{=} \frac{-e^2}{(2\pi)^3 2k} \int d^4X e^{ik(t-z)} \langle J^\mu(0) J_\mu(X) \rangle$$

- This is a < Wightman function, $\Pi^<(K)$. We could in principle just take these propagators and Feynman rules and compute. However
  - this basis is not optimal, and we have not discussed cutting rules yet
  - we will rapidly run into failures of the loop expansion