Energy density of the free boson gas

$$e := \langle \Theta_{00} \rangle = \langle \partial_0 \varphi(x) \partial_0 \varphi(x) \rangle - \frac{1}{2} \langle \partial_e \varphi \partial_e \varphi \rangle \Big\} = T \sum_m \int \frac{d^3p}{(2\pi)^3} \frac{\omega_m^2 - \overbrace{(\omega_m^2+p^2)}^{=0 \text{ in dim. reg.}}/2}{\omega_m^2 + p^2}$$

in DR (dim. reg.)

$$\int \frac{d^dp}{(2\pi)^d} \frac{1}{\omega_m^2+p^2} = \frac{1}{(4\pi)^{d/2}} \Gamma(1-d/2) (\omega_m^2)^{d/2-1} \overset{d=3}{=} \frac{|\omega_m|}{4\pi}$$

$$\Rightarrow \quad e = \frac{1}{4\pi} T \sum_m (2\pi T)^3 |m|^3 = \frac{8\pi^3 T^4}{4\pi} 2\zeta(-3) \qquad \left[\sum_{m=1}^{\infty} \frac{1}{m^a} = \zeta(a) \right]$$

$$= 4\pi^2 T^4 \zeta(-3) = \frac{\pi^2 T^4}{30}.$$

Stefan-Boltzmann pressure for a free bosonic scalar gas.

But what happened? Where is the physics?! Answer later

Remark: For a more familiar form, use $\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}$ for $m \in N$

$$\zeta(2m) = (-1)^{m+1} \frac{B_{2m} (2\pi)^{2m}}{2(2m)!} \quad \text{for } m \in N$$

$$\Rightarrow \quad \zeta(-3) = \frac{-B_4}{4}, \quad \zeta(4) = -\frac{B_4 (2\pi)^4}{4 \cdot 24}$$

$$\Rightarrow \quad \zeta(-3) = \frac{24}{16\pi^4} \zeta(4)$$

In the real-time formalism

$$e = \Big[\overbrace{\langle \partial^0 \varphi(x) \partial^0 (\varphi(x)) \rangle - \frac{1}{2} \langle \partial^\mu \varphi \partial_\mu \varphi \rangle}^{\text{Wightman fncts.}} \Big]$$

$\langle \partial^\mu \varphi \partial_\mu \varphi \rangle \to \propto (p_0^2 - p^2)$ vanishes

for a free massless boson $e(P) := 2\pi\, \varepsilon(p^0)\, \delta(p_0^2 - p^2)$

$D = d+1 = 4 - 2\varepsilon$ $\qquad n_B(-P) = -(1 + n_B(P))$

$$\to \quad e = \int \frac{d^Dp}{(2\pi)^D} p^2 \left(1 + n_B(p^0)\right) 2\pi\, \varepsilon(p^0)\, \delta(p_0^2 - p^2) = \int \frac{d^dp}{(2\pi)^d} \frac{p^2}{2p} \left[\cancel{1} + n_B(p) - \cancel{1} - n_B(-p) \right]$$

$$= \int \frac{d^dp}{(2\pi)^d} p \Big[\underbrace{\frac{1}{2}}_{\text{vacuum fluctuation (killed by DR)}} + \overbrace{n_B(p)}^{\text{thermal fluctuation}} \Big] = \int \frac{d^dp}{(2\pi)^d} p\, n_B(p) = \frac{\pi^2 T^4}{30}$$

For the last integral, recall that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \quad \Rightarrow \quad \int_0^\infty dp\, p^3 n_B(p) = T^4 \int_0^\infty dx \frac{x^3}{e^x - 1} = T^4 \Gamma(4) \zeta(4) = \frac{6 \pi^4 T^4}{90} = \frac{\pi^4 T^4}{15}$$

Not the way thermodynamics is usually computed (see Laine Vuorinen, $p = \frac{\partial}{\partial V} T \ln Z$)