Lattice QCD in extreme conditions

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CRC–TR 211
Strong-interaction matter under extreme conditions

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Outline overall

- lecture 1: introduction to QCD and thermodynamics
- lecture 2: hot Yang-Mills theory and hot QCD on the lattice
- lecture 3: QCD in extreme conditions on the lattice

Outline lecture 2

- some elements of finite temperature field theory
- lattice discretization of the QCD action
- confinement and deconfinement
- Polyakov loop and center clusters
- equation of state

Finite temperature field theory

Equation of state

- free energy (density)

$$F = -T \log \mathcal{Z} \qquad f = \frac{F}{V}$$

- entropy density

$$s = -\frac{1}{V}\frac{\partial F}{\partial T}$$

- pressure

$$p = -\frac{\partial F}{\partial V} \xrightarrow{V\to\infty} -f$$

- energy density

$$\epsilon = -\frac{1}{V}\frac{\partial \log \mathcal{Z}}{\partial(1/T)} = f + Ts$$

- interaction measure

$$I = \mathrm{tr}\, T_{\mu\nu} = \epsilon - 3p$$

- all we need is $\log \mathcal{Z}$

Scalar quantum field theory

- partition function for real scalars

$$\mathcal{Z} = \int_{\phi(x,0)=\phi(x,1/T)} \mathcal{D}\phi\, e^{-S[\phi]}$$

  over commuting numbers $\phi(x,\tau)$

- partition function for complex scalars

$$\mathcal{Z} = \int_{\substack{\phi(x,0)=\phi(x,1/T)\\ \phi^*(x,0)=\phi^*(x,1/T)}} \mathcal{D}\phi^*\, \mathcal{D}\phi\, e^{-S[\phi^*,\phi]}$$

- for quadratic actions (free case)

$$\int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi M \phi} = C \cdot [\det(M)]^{-1/2}$$

$$\int \mathcal{D}\phi^* \mathcal{D}\phi\, e^{-\phi^* M \phi} = C' \cdot [\det(M)]^{-1}$$

Fermionic quantum field theory

- partition function for fermions

$$\mathcal{Z} = \int_{\psi(x,0)=-\psi(x,1/T)} \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-S[\bar{\psi},\psi]}$$

  over Grassmann numbers $\psi(x,\tau)$
- for bilinear actions (not just free case!)

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-\bar{\psi}M\psi} = C'' \cdot \det(M)$$

- Euclidean Dirac operator

$$M = \not{\partial} + m = \gamma_\mu \partial_\mu + m \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$$

Gauge field theory

- gauge field A_μ $(\mathrm{U}(1))$ or A_μ^a with $a = 1 \ldots N_c^2 - 1$ $(\mathrm{SU}(N_c))$
- partition function

$$\mathcal{Z} = \int_{A_\mu^a(x,0)=A_\mu^a(x,1/T)} \mathcal{D}A_\mu^a \, e^{-S[A_\mu^a]}$$

- note 1: this is gauge invariant, but to derive it we needed to fix a gauge
- note 2: to evaluate $\mathcal{Z}$ via perturbation theory, we need to fix a gauge again
- free case

$$f_{\mathrm{U}(1)} = f_{\text{real scalar}} \cdot (4-2) \qquad f_{\mathrm{SU}(N_c)} = f_{\text{real scalar}} \cdot (4-2) \cdot (N_c^2 - 1)$$

ghosts cancel half the gauge field contribution

Remark: ultraviolet divergences

- vacuum contribution to free energy is UV divergent

$$f_{\mathrm{vac}} = \frac{1}{2} \int_0^{\Lambda} \frac{4\pi p^2 \mathrm{d}p}{(2\pi)^3} \sqrt{p^2 + m^2} = \underline{\mathcal{O}(\Lambda^4) + \mathcal{O}(m^2\Lambda^2) + \mathcal{O}(m^4 \log \Lambda^2) + \mathcal{O}(m^4)}$$

- renormalize free energy at $T = 0$ to zero

$$f^r = f - f(T = 0)$$

(normal ordering in operator language)

- for the equation of state

$$p^r = p - p(T = 0) \qquad \epsilon^r = \epsilon - \epsilon(T = 0)$$

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$$p^r = p - p(T=0) \qquad \epsilon^r = \epsilon - \epsilon(T=0)$$

  no renormalization necessary for entropy $s = -\partial f/\partial T$

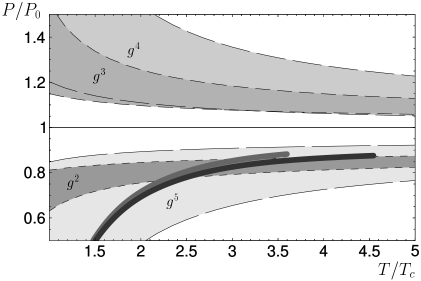
Perturbation theory

- pure glue free pressure

$$p^r_{\mathrm{SU}(3)} = -f^r_{\mathrm{SU}(3)} = -16 \cdot \int \frac{d^3\mathbf{p}}{(2\pi)^3} T \log(1 - e^{-|\mathbf{p}|/T}) = \frac{8\pi^2}{45} \cdot T^4$$

- perturbation theory in g (vertices A^3 and A^4)

  Blaizot, Iancu, Rebhan PRD '03



bands: renormalization scale depence $g^2(\pi T \leq \bar{\mu} \leq 4\pi T)$

- need for lattice gauge theory

QCD on the lattice

In the continuum

- Euclidean path integral

$$\mathcal{Z} = \int \mathcal{D}A_\mu\, \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \exp\left[-\int_0^{1/T} \mathrm{d}x_4 \int_{L^3} \mathrm{d}^3x\, \mathcal{L}(x)\right]$$

- Lagrangian

$$\mathcal{L} = \sum_f \bar{\psi}_f(\not{D} + m_f)\psi_f + \frac{1}{2g^2}\mathrm{Tr}\, G_{\mu\nu} G_{\mu\nu}$$

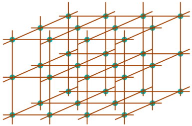
- fields

$$A_\mu = A_\mu^a T_{cd}^a, \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu + [A_\mu, A_\nu]$$
$$\psi_f = \psi_{f,\alpha,c}, \quad \not{D}_{cd}^{\alpha\beta} = \gamma_\mu^{\alpha\beta}(\partial_\mu + iA_\mu^a T_{cd}^a)$$

- parameters: m_f quark masses and g strong coupling

Discretization

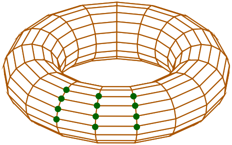
- discretize space $L = a \cdot N_s$ and imaginary time $1/T = a \cdot N_t$
  $x = a \cdot n$



- path integral

$$\mathcal{Z} = \lim_{\substack{a\to 0\\ N_t\to\infty\\ N_s\to\infty}} \int \prod_{n,\mu} \mathrm{d}A_\mu(n)\, \mathrm{d}\bar{\psi}(n)\, \mathrm{d}\psi(n)\, \exp\left[-S^{\text{lat}}_{F+G}\right]$$

- nonzero T: bosons periodic, fermions antiper. in imag. time
- finite volume: periodic boundary conditions in space



Lattice action

- variable substitution to parallel transporter

$$U_\mu = \exp(iaA_\mu) \in \mathrm{SU}(3)$$

- symbolically

$$U_\mu(n) = \underset{n \quad \mu}{\longrightarrow} \qquad U_\mu^\dagger(n-\hat{\mu}) = \underset{\mu \quad n}{\longleftarrow}$$

- gauge transformation (Ω unitary)

$$A_\mu(n) \to \Omega(n) A_\mu(n) \Omega^\dagger(n) + i(\partial_\mu \Omega(n))\Omega^\dagger(n)$$
$$U_\mu(n) \to \Omega(n) U_\mu(n) \Omega^\dagger(n+\hat{\mu})$$

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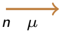
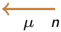
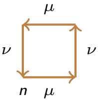
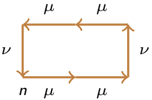
$U_\mu(n) =$ 

$U_\mu^\dagger(n - \hat{\mu}) =$ 

- smallest closed (=gauge invariant) curves

$\mathrm{Tr}\, P_{\mu\nu}(n) =$ 

$\mathrm{Tr}\, P_{\mu\nu}^{1\times 2}(n) =$ 

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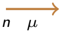
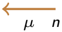
$U_\mu(n) =$ 

$U_\mu^\dagger(n - \hat{\mu}) =$ 

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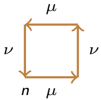
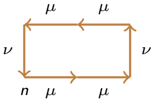
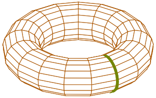
$\mathrm{Tr}\, P_{\mu\nu}(n) =$ 

$\mathrm{Tr}\, P_{\mu\nu}^{1\times 2}(n) =$ 

or winding loops (Polyakov loop)

$\mathrm{Tr}\, L_\mu(n) =$ 



Gluon action

- expansion towards continuum limit

$$\mathrm{Re}\,\mathrm{Tr}(\mathbb{1} - P_{\mu\nu}(n)) = \frac{a^4}{2}\mathrm{Tr}\, G_{\mu\nu}(n) G_{\mu\nu}(n) + \mathcal{O}(a^6)$$

- full action

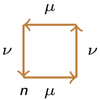
$$S_G^{\mathrm{lat}} = \frac{\beta}{3}\sum_n \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{Tr}(\mathbb{1} - P_{\mu\nu}(n))$$

- strong coupling parameter

$$\beta = \frac{6}{g^2}$$

- path integral over links

$$\mathcal{Z} = \int \mathcal{D}U \, \exp\Bigg[-\beta \cdot \frac{1}{3} \sum_{n,\mu<\nu} \mathrm{Re}\Bigg(3 -$$



$$\Bigg)\Bigg]$$

Continuum limit

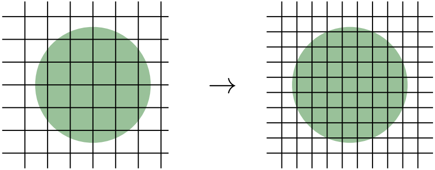
- continuum RG equation perturbatively

$$\bar{\mu}\frac{\partial g_R(\bar{\mu})}{\partial\bar{\mu}} = -b_1 g_R^3(\bar{\mu}) + \dots$$

- lattice RG equation perturbatively

$$a^{-1}\frac{\partial g(a^{-1})}{\partial(a^{-1})} = -b_1 g^3(a^{-1}) + \dots$$

- asymptotic freedom: cont. limit at $g \to 0$, $\beta = 6/g^2 \to \infty$
- continuum limit nonperturbatively



- lattice spacing from scale setting $\boxed{a(\beta)}$

Finite temperature on the lattice

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- spatial size: $L = N_s a(\beta)$
- temporal size: $1/T = N_t a(\beta)$

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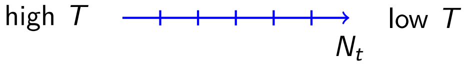
- temporal size: $1/T = N_t a(\beta)$
- fixed-β approach: change T by changing N_t
  - only discrete temperatures possible $\times$
  - all temperatures have same lattice spacing ✓
  - scale setting and renormalization only once ✓



Finite temperature on the lattice

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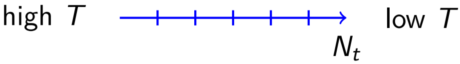
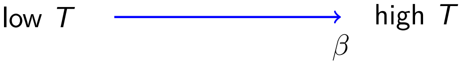


- fixed-N_t approach: change T by changing β
  - continuous temperatures possible ✓
  - different temperatures have different lattice spacing ×
  - scale setting and renormalization for each β ×



Confinement and deconfinement

Confinement

- free energy of thermal medium: $F = -T \log \mathcal{Z}$
  free energy in presence of static quark: $F_q = -T \log \mathcal{Z}_q$
- confinement: $F_q - F = \infty$
  deconfinement: $F_q - F < \infty$
- operator language

$$e^{-(F_q - F)/T} = \frac{\mathcal{Z}_q}{\mathcal{Z}} = \frac{\mathrm{tr}\, e^{-\hat{H}/T}\, \hat{P}_q}{\mathrm{tr}\, e^{-\hat{H}/T}}$$

  with P_q projector to states including static quark Ukawa '93
- with path integral

$$P(\mathbf{x}) = \mathrm{tr}\, \mathcal{P} \exp\left(i \int_0^{1/T} \mathrm{d}x_4 A_4(x) \right)$$

Polyakov loop

- on the lattice

$$P(\mathbf{n}) = \mathrm{tr} \prod_{n_4=0}^{N_t-1} U_t(n) =$$

- average Polyakov loop $P = \frac{1}{V} \sum_{\mathbf{n}} P(\mathbf{n})$
- order parameter for confinement

$$\langle P \rangle = e^{-(F_q - F)/T} \begin{cases} = 0 & \text{confinement} \\ \neq 0 & \text{deconfinement} \end{cases}$$

- static quark and antiquark at separation r

$$e^{-(F_{q\bar{q}}(r) - F)/T} = \langle P(0) P^\dagger(r) \rangle \xrightarrow{r \to \infty} \langle P \rangle \langle P^\dagger \rangle + \mathcal{O}(e^{-\sigma r/T})$$

so

$$F_{q\bar{q}}(r \to \infty) - F \propto \begin{cases} \sigma r & \text{confinement} \\ \text{const.} & \text{deconfinement} \end{cases}$$

Center symmetry

- symmetry corresponding to confinement/deconfinement?

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- transformation under which S is invariant but P is not

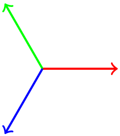
  $\Rightarrow$ center transformation

- elements of center of group

$$V_\zeta \quad \text{so that} \quad [V_\zeta, U] = 0 \quad \forall\, U \in \mathrm{SU}(3)$$

in our case it is

$$\mathbb{Z}(3) = \{\mathbb{1}, e^{i2\pi/3}\mathbb{1}, e^{-i2\pi/3}\mathbb{1}\} = \zeta \cdot \mathbb{1}$$

$\zeta =$ 

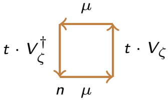
Center symmetry

- center transformation

$$U_t(\mathbf{n}, \bar{n}_4) \to V_\zeta \cdot U_t(\mathbf{n}, \bar{n}_4) \qquad V_\zeta \in \mathbb{Z}(3) \qquad \forall\ \mathbf{n}$$

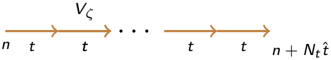
- gauge action is invariant

$$[V_\zeta, U_\mu(n)] = 0 \quad \Rightarrow \quad S_G \to S_G$$



- Polyakov loop not invariant

$$P(\mathbf{n}) \to \zeta \cdot P(\mathbf{n})$$

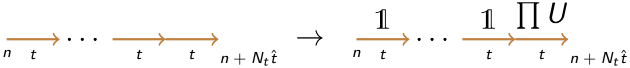


confinement	$\langle P \rangle = 0$	$\mathrm{Z}(3)$ symmetry intact
deconfinement	$\langle P \rangle \neq 0$	$\mathrm{Z}(3)$ symmetry spontaneously broken

Spontaneous center symmetry breaking

- Polyakov loop in Polyakov gauge

$P(\mathbf{n}) =$



  is a boundary condition
- if $P(\mathbf{n})$ is constant, it can be diagonalized

$$P = \operatorname{tr}\operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{-i(\varphi_1+\varphi_2)})$$

- perturbative treatment at high temperature:
  $\log \mathcal{Z}(\varphi_1, \varphi_2)$ in the background of a constant Polyakov loop
  🔗 Roberge, Weiss NPB '86
- three degenerate minima at

$$\varphi_1 = \varphi_2 = 0 \qquad \varphi_1 = \varphi_2 = 2\pi/3 \qquad \varphi_1 = \varphi_2 = -2\pi/3$$

$$P = 3 \qquad P = 3\,e^{2\pi i/3} \qquad P = 3\,e^{-2\pi i/3}$$

Spontaneous center symmetry breaking

- from perturbation theory we except three minima

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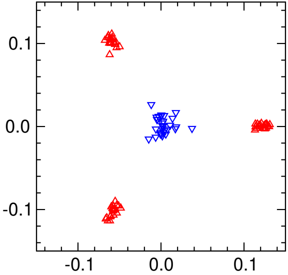
- scatter plot at low T and high T Danzer et al. JHEP '08



- remember Ising model recipe: $\lim_{V\to\infty} \langle |M| \rangle$ at $h = 0$

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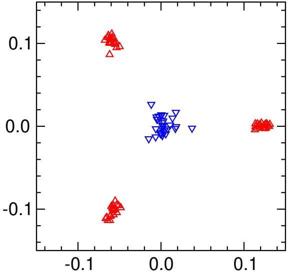

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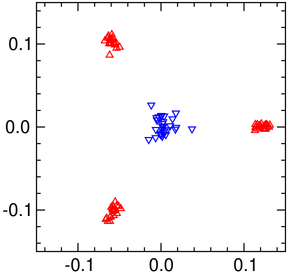


- remember Ising model recipe: $\lim_{V\to\infty} \langle |M| \rangle$ at $h = 0$
- alternative 1: measure $\langle |P| \rangle$
- alternative 2: measure $\langle P_{\mathrm{rot}} \rangle$ with rotated Polyakov loop

$$-\pi/3 < \arg P_{\mathrm{rot}} < \pi/3$$

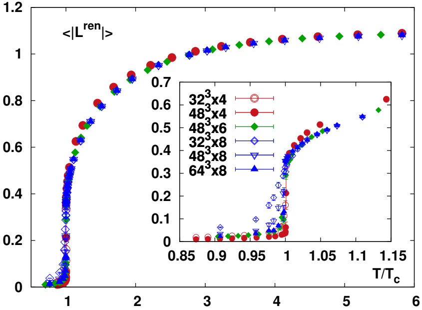
Dictionary 1.

	Ising model	Yang-Mills theory
symmetry group	Z(2)	Z(3)
spontaneous breaking	$\langle M \rangle = \frac{\partial \log \mathcal{Z}}{\partial h}$	$\langle P \rangle$
explicit breaking	h	?
symmetry restoration	at high T	at low T

Deconfinement transition

Results: Polyakov loop

- $\langle|P|\rangle$ as function of T in the fixed N_t-approach
  Lo et al. PRD '13



- note: UV renormalization: F_q additive $\rightsquigarrow$ $\langle P\rangle$ multiplicative

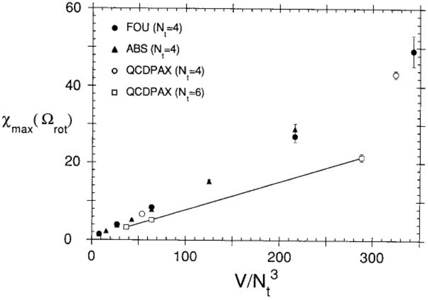
Results: susceptibility

- susceptibility of order parameter

$$\chi_P = \langle P_{\rm rot}^2 \rangle - \langle P_{\rm rot} \rangle^2$$

- how does peak height scale with volume?

  Iwasaki et al. PRD '92



- $\chi_P(L, T_c(L)) \propto L^3 = V$ first-order phase transition

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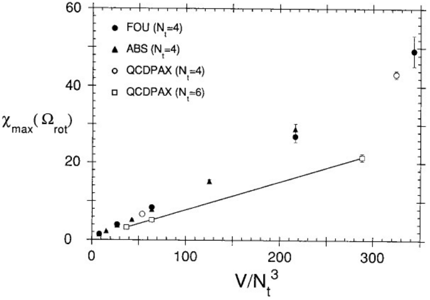

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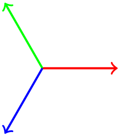
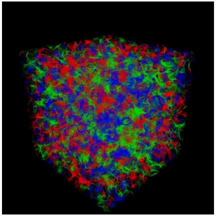
Center clusters

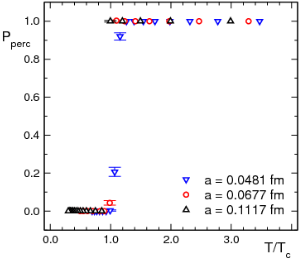
Center clusters

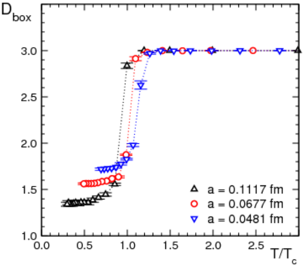
- local distribution of $P(\mathbf{n})$ Stokes, Kamleh, Leinweber Ann. Phys. '14
  https://www.youtube.com/watch?v=T4sRON6uOz0





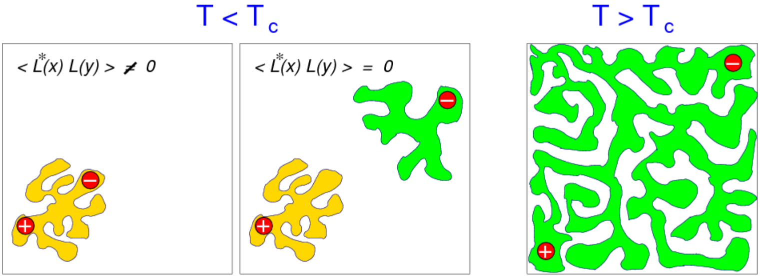
- clusters percolate at T_c and they are fractals
  Endrődi, Gattringer, Schadler PRD '14





Center clusters

- insight into confinement and deconfinement mechanism
  Gattringer, Schmidt JHEP '10



$$\left\langle P(0)P^{\dagger}(r)\right\rangle \propto \exp(-\sigma r/T)$$

Equation of state

Equation of state, reminder

- free energy (density)

$$F = -T \log \mathcal{Z} \qquad f = \frac{F}{V}$$

- entropy density

$$s = -\frac{1}{V}\frac{\partial F}{\partial T}$$

- pressure

$$p = -\frac{\partial F}{\partial V} \xrightarrow{V\to\infty} -f$$

- energy density

$$\epsilon = -\frac{1}{V}\frac{\partial \log \mathcal{Z}}{\partial(1/T)} = f + Ts$$

- interaction measure / trace anomaly

$$I = \mathrm{tr}\, T_{\mu\nu} = \epsilon - 3p$$

EoS - methods

- very high T: perturbation theory, Hard Thermal Loop resummation

  Braaten, Pisarski PRL '90   Andersen, Strickland, Su JHEP '10
- low T: glueball resonance gas model
- intermediate T: lattice gauge theory
- how to determine $\log \mathcal{Z}$ via expectation values?
  - derivative method
  - integral method
  - moving frame method

Derivative method

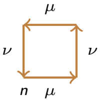
Derivative method

- trace anomaly as a derivative

$$\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}\log a} = a\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}a} = \frac{1}{T}\frac{\partial\log\mathcal{Z}}{\partial(1/T)} + 3L^3\frac{\partial\log\mathcal{Z}}{\partial(L^3)} = -\frac{V}{T}(\epsilon - 3p)$$

- how does $\log\mathcal{Z}$ depend on a?

$$\mathcal{Z} = \int \mathcal{D}U \exp\Big[-\beta\cdot\overbrace{\frac{1}{3}\sum_{n,\mu<\nu}\mathrm{Re}\Big(3 - \square\Big)}^{S_G}\Big]$$



Derivative method

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- implicitly via β

$$\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}\log a} = \frac{\partial\log\mathcal{Z}}{\partial\beta}\cdot\frac{\partial\beta}{\partial\log a} = \langle -S_G\rangle\cdot\frac{a(\beta)}{a'(\beta)}$$

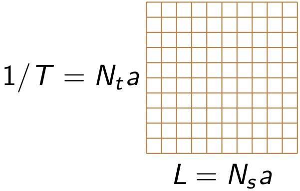
  for which we need to know $a(\beta)$ from scale setting

- so

$$I = \epsilon - 3p = \frac{T}{V}\langle S_G\rangle\frac{a(\beta)}{a'(\beta)}$$

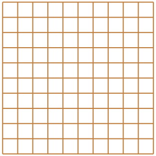
Derivative method

- how to get just p or just ϵ?

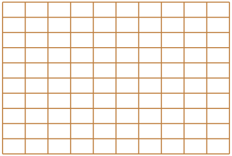


Derivative method

- how to get just p or just ϵ?

$1/T = N_t a$ 

$L = N_s a$

$1/T = N_t a_t$ 

$L = N_s a$

- anisotropic lattice $\xi = a/a_t$

$$\epsilon = -\frac{T}{V}\left.\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}\log a_t}\right|_a \qquad 3p = \frac{T}{V}\left.\frac{\mathrm{d}\log\mathcal{Z}}{\mathrm{d}\log a}\right|_{a_t}$$

these can be evaluated at $\xi = 1$

Karsch NPB '82   Engels et al. NPB '82

Anisotropy coefficients

- anisotropic lattice action

$$S_G = \overbrace{\xi_0 \cdot \frac{1}{3} \sum_{n,\mu \neq t} \mathrm{Re}\left(3 - {}_{t}\underset{n\;\;\mu}{\overset{\mu}{\square}}{}_{t}\right)}^{S_G^t} + \overbrace{\frac{1}{\xi_0} \cdot \frac{1}{3} \sum_{\substack{n,\mu<\nu \\ \mu,\nu \neq t}} \mathrm{Re}\left(3 - {}_{\nu}\underset{n\;\;\mu}{\overset{\mu}{\square}}{}_{\nu}\right)}^{S_G^s}$$

and we need one more scale setting relation $\boxed{\xi(\beta, \xi_0)}$

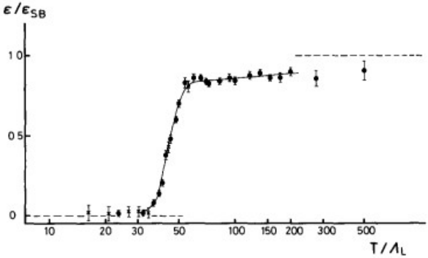
- energy density

$$\epsilon = -\xi^2 \frac{T}{V} \left[\langle S_G^s \rangle \frac{\partial(\beta \xi_0)}{\partial \xi} + \langle S_G^t \rangle \frac{\partial(\beta/\xi_0)}{\partial \xi} \right]$$

- anisotropy coefficients difficult to measure precisely

Results: derivative method

- results for energy density Engels et al. NPB '82



- remember additive divergences in $\log \mathcal{Z} \propto a^{-4}$
- renormalized energy density

$$\epsilon^{\mathrm{r}} = \epsilon - \epsilon(T \approx 0)$$

involving cancellation of $\mathcal{O}(a^{-4})$ divergences

$$\left\langle S_G^{t,s} \right\rangle_{N_s^3 N_t} - \left\langle S_G^{t,s} \right\rangle_{N_s^4}$$

Integral method

Integral method

- integrate back the derivatives Boyd et al. NPB '96

$$\log \mathcal{Z}(\beta_1) - \log \mathcal{Z}(\beta_0) = \int_{\beta_0}^{\beta_1} \mathrm{d}\beta \, \frac{\partial \log \mathcal{Z}}{\partial \beta}$$

- works in the fixed N_t-approach
- differences of dimensionless pressures

$$p(T_1)a_1^4 - p(T_0)a_0^4 = -\frac{1}{N_s^3 N_t} \int_{\beta_0}^{\beta_1} \mathrm{d}\beta \, \langle S_G \rangle$$

or

$$\frac{p(T_1)}{T_1^4} - \frac{p(T_0)}{T_0^4} = -\frac{N_t^3}{N_s^3} \int_{\beta_0}^{\beta_1} \mathrm{d}\beta \, \langle S_G \rangle$$

- is this UV finite?

Renormalization

- $p(T_1, a) - p(T_0, a)$ UV finite but $p(T_1, a_1) - p(T_0, a_0)$ divergent
- need to do $T \approx 0$ subtraction

$$\frac{p^{\mathrm{r}}(T_1)}{T_1^4} - \frac{p^{\mathrm{r}}(T_0)}{T_0^4} = -\frac{N_t^3}{N_s^3} \int_{\beta_0}^{\beta_1} \mathrm{d}\beta \left[\langle S_G \rangle_{N_s^3 N_t} - \langle S_G \rangle_{N_s^4} \right]$$

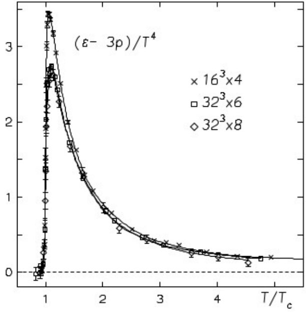
with starting point β_0 where $p^{\mathrm{r}}(T_0)/T_0^4 \approx 0$

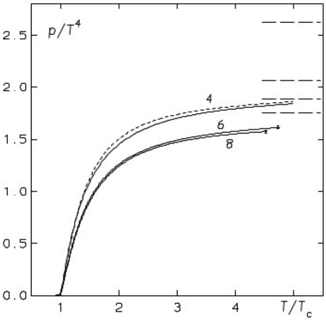
- renormalized interaction measure

$$\frac{I^{\mathrm{r}}}{T^4} = \frac{N_t^3}{N_s^3} \frac{a(\beta)}{a'(\beta)} \left[\langle S_G \rangle_{N_s^3 N_t} - \langle S_G \rangle_{N_s^4} \right]$$

Results: integral method

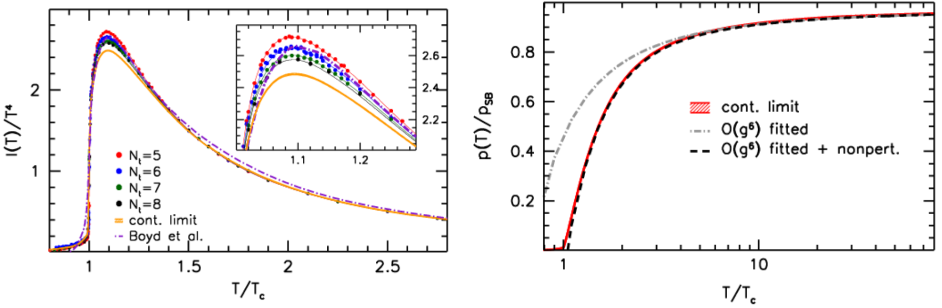
- need interpolation ($+ a(\beta)$) for I, then numerical integral for p
  Boyd et al. NPB '96





Results: integral method

- need interpolation $(+\ a(\beta))$ for I, then numerical integral for p
  Boyd et al. NPB '96
- update on finer lattices Borsányi, Endrődi et al. JHEP '12



Moving frame method

Thermal medium at relativistic speeds

- so far we have been in the rest frame

$$\left\langle \hat{\Theta}_{\mu\nu} \right\rangle = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- now consider frame moving with $\mathbf{v} = (v, 0, 0)$

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xkcd.com/1233

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- now consider frame moving with $\mathbf{v} = (v, 0, 0)$

$$\left\langle \hat{\Theta}'_{\mu\nu} \right\rangle = \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu} \left\langle \hat{\Theta}_{\rho\sigma} \right\rangle = \begin{pmatrix} \frac{\epsilon+v^2 p}{1-v^2} & v\frac{\epsilon+p}{1-v^2} & 0 & 0 \\ v\frac{\epsilon+p}{1-v^2} & \frac{p+v^2\epsilon}{1-v^2} & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- recall $\epsilon = f + Ts = -p + Ts$
  therefore

$$\left\langle \hat{\Theta}_{01} \right\rangle_{v} = \frac{v}{1-v^2}(\epsilon + p) = \frac{v}{1-v^2} Ts$$

Shifted boundary conditions

- partition function becomes

  Giusti, Meyer PRL '11   Giusti, Meyer JHEP '13

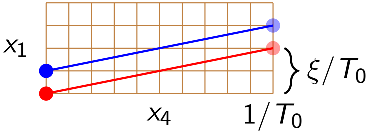
$$\mathcal{Z} = \text{tr} \exp\left[-(\hat{\Theta}_{00} - v\hat{\Theta}_{01})/T_0\right]$$

in Euclidean space $v = i\xi$

$$\mathcal{Z} = \text{tr} \exp\left[-(\hat{\Theta}_{00} - i\xi\hat{\Theta}_{01})/T_0\right]$$

- states with x_1-momentum Θ_{01} weighted by $e^{i\xi\Theta_{01}/T_0}$
  $\Rightarrow$ shifted boundary conditions

$$U(n_1, 0) = U(n_1 + \xi/T_0, 1/T_0)$$



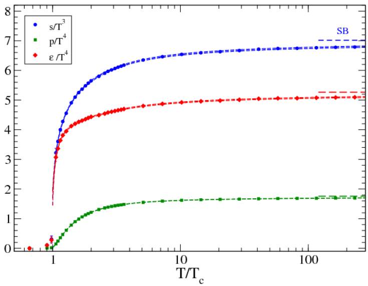
- but watch out: $1/T = 1/T_0 \cdot \sqrt{1+\xi^2}$

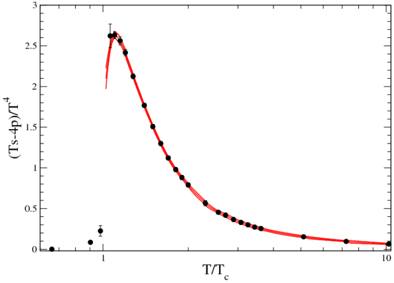
Results: moving frame method

- entropy in moving frame

$$Ts = \frac{1+\xi^2}{\xi} \left\langle \hat{\Theta}_{01} \right\rangle_\xi \cdot Z_T(a)$$

- simulations with $\xi = \{1, \sqrt{2}, \sqrt{3}\} \cdot a$ Giusti, Pepe JHEP '16
- multiplicative renormalization Z_T for operator $\Theta_{\mu\nu}$
- recover full EoS from s





Equation of state: summary

- derivative method
  - works with a single ensemble ✓
  - needs anisotropy coefficients ×
- moving frame method
  - works with a single ensemble ✓
  - needs renormalization constants ×
- integral method: most powerful up to date
  - only simple expectation values required ✓
  - needs many ensembles ×
- Jarzynski's method 🔗 Caselle et al. PRD '18
  and other approaches