Recent finite-volume developments in QED_L

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Overview

- Motivation: Precision tests of the Standard Model
- Isospin-breaking needed $\implies$ Simulate Lattice QCD$+$QED
- Several ways to have QED in finite volume: $\mathrm{QED_L}$
- My goal today:
  1. What is $\mathrm{QED_L}$?
  2. What about finite-size effects in the simulations?
- Reaching the infinite-volume limit:
  1. Analytical correction for finite-size effects: Loop calculations
  2. Simulations at different volumes: Fits

Motivation

- Top row unitarity of CKM matrix: Study $\frac{|V_{us}|}{|V_{ud}|}$
- Leptonic decays: $P^- \to \ell^- \nu_\ell[\gamma]$ for $P = \pi, K$

$$\Gamma\left(P^- \to \ell^- \nu_\ell[\gamma]\right) = \Gamma^{\mathrm{tree}}\left[1 + \delta R_P\right]$$

$$\Gamma^{\mathrm{tree}} = \frac{G_F^2}{8\pi}|V_{ij}|^2 f_P^2\, m_P m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2$$

- δR_P: $\alpha \neq 0$ and $m_u \neq m_d$
- Ratios of decays: $K_{\ell 2}/\pi_{\ell 2} \longrightarrow$ combine experiment and theory

$$\frac{|V_{us}|^2}{|V_{ud}|^2} = \frac{\Gamma\left(K^- \to \ell^- \nu_\ell[\gamma]\right)}{\Gamma\left(\pi^- \to \ell^- \nu_\ell[\gamma]\right)} \frac{m_{K^-}^3}{m_{\pi^-}^3} \frac{\left(m_{\pi^-}^2 - m_{\mu^-}^2\right)^2}{\left(m_{K^-}^2 - m_{\mu^-}^2\right)^2} \frac{(f_\pi/f_K)^2}{1 + \delta R_K - \delta R_\pi}$$

- Obtain theory part from lattice: Focus on $\delta R_{K\pi} = \delta R_K - \delta R_\pi$

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Our calculation

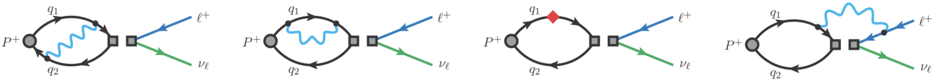
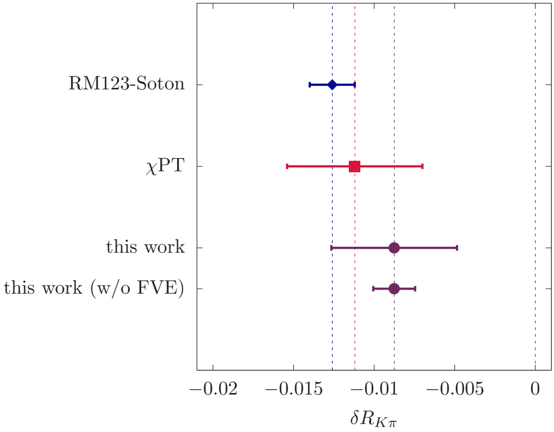


Figure from M. Di Carlo, Lattice 2022

	RBC/UKQCD, 2023
Masses	Phys. point simulation
Fermion action	Domain Wall
Continuum limit	Single lattice spacing
QED prescription	$\mathrm{QED_L}$
Sea-quark effects	Electro-quenched
Infinite-volume limit	Single volume

Single volume: $L/a = 48$, correct $\delta R_{K\pi}$ analytically

The final result



Our work [RBC/UKQCD 23]:

$\delta R_{K\pi} = -0.0086(39)$

total	(39)
total (w/o FVE)	(13)
statistical	(3)
FVE	(37)
fit	(11)
QED quenching	(5)
discretisation	(5)

$\chi\text{PT}: \ \delta R_{K\pi} = -0.0112(21)$

$\text{RM123/Soton}: \ \delta R_{K\pi} = -0.0126(14)$

NB: New theoretical approach beyond QED_L suggested [Feng et al., 2023]

QED in a finite volume

- Gauss' law: Difficult to define charged states in finite volume with periodic boundary conditions
(photon zero-momentum modes and absence of mass gap)
- Several prescriptions
  1. $\mathrm{QED_C}$: Charge-conjugated boundary conditions
  [Kronfeld, Wiese 1991–1993; RC* 2019]
  2. $\mathrm{QED_M}$: Photon mass m_γ
  [Endres, Shindler, Tiburzi, Walker-Loud 2016; Bussone, Della Morte, Janowski 2018]
  3. $\mathrm{QED_\infty}$: Do the QED part in infinite volume
  [Feng, Jin 2018]
  4. $\mathrm{QED_L}$: Exclude photon zero-mode on each time-slice
  [Hayakawa, Uno 2008]

$$\mathrm{QED_L} : \quad \sum_{\mathbf{k}\in\Pi} \longrightarrow \sum_{\mathbf{k}}{}' = \sum_{\mathbf{k}\neq 0}$$

- Each has advantages/drawbacks : $\mathrm{QED_L}$ simple but non-local

Finite-size effects

- Massless photon + no zero-mode ($\mathrm{QED_L}$ and $\mathrm{QED_C}$)

  $V = \mathbb{R} \times L^3$: $\implies$ Finite-size effects (FSEs) in observable $\mathcal{O}(L)$:

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}_{\mathrm{IV}} = \kappa_0 + \kappa_{\log} \log m_P L + \kappa_1 \frac{1}{m_P L} + \kappa_2 \frac{1}{(m_P L)^2} + \dots$$

- Scaling in L is observable-dependent:
  e.g. self-energy $\kappa_0 = \kappa_{\log} = 0$
- Coefficients depend on physical particle properties: masses, charges, structure (form-factors):
  Point-like + structure-dependent
- NB: Coefficients are prescription-dependent!

[Davoudi, Savage 2014; BMW 2015; RM-123/Soton 2017; Davoudi, Harrison, Jüttner, Portelli, Savage 2019; Bijnens, Harrison, H-T, Janowski, Jüttner, Portelli 2019; Di Carlo, Hansen, H-T, Portelli 2021]

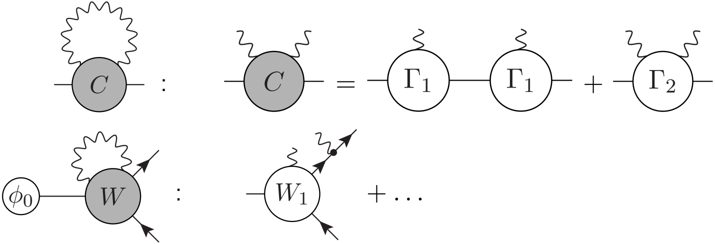
- $\mathrm{QED_M}$ and $\mathrm{QED_\infty}$: no power-law effects

Finite-size effects

- Consider IR-safe quantity $\mathcal{O}$ at order α
- Let $k = (k_0, \mathbf{k})$ be the photon momentum
- Finite-size effects in $\mathcal{O}(L)$ given by:

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}_{\mathrm{IV}} = \left(\frac{1}{L^3} \sum_{\mathbf{k}\in\Pi} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, \mathbf{k}\right)}{k^2 \left[(p-k)^2 + m^2\right]}$$

- The function $f_{\mathcal{O}}\left(k_0, \mathbf{k}\right)$ has no poles: Depends on observable and is structure-dependent
- Examples: Self-energy or leptonic decays



Finite-size effects

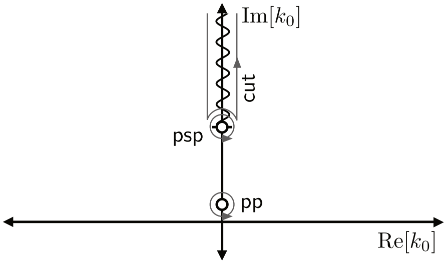
- Let us first consider the k_0 integral

$$g_{\mathcal{O}}(\mathbf{k}) = \int \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}(k_0, \mathbf{k})}{k^2\left[(p-k)^2 + m^2\right]}$$

- 2 poles and residual analytical structure (cut)

$$g_{\mathcal{O}}(\mathbf{k}) = \int_{\text{poles}} \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}(k_0, \mathbf{k})}{k^2\left[(p-k)^2 + m^2\right]} + \int_{\text{rest}} \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}(k_0, \mathbf{k})}{k^2\left[(p-k)^2 + m^2\right]}$$

Example: Self-energy



Finite-size effects

- Once we have determined $g_{\mathcal{O}}(\mathbf{k}) = g_{\mathcal{O}}^{\text{poles}}(\mathbf{k}) + g_{\mathcal{O}}^{\text{rest}}(\mathbf{k})$ we have

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\in\Pi} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \left[g_{\mathcal{O}}^{\text{poles}}(\mathbf{k}) + g_{\mathcal{O}}^{\text{rest}}(\mathbf{k}) \right]$$

- Soft photons: Expand in small $|\mathbf{k}| = \frac{2\pi|\mathbf{n}|}{L} \implies$ expansion in L

$$\Delta\mathcal{O}(L) = \kappa_0 + \kappa_{\log} \log m_P L + \kappa_1 \frac{1}{m_P L} + \kappa_2 \frac{1}{(m_P L)^2} + \kappa_3 \frac{1}{(m_P L)^3} \dots$$

- $g_{\mathcal{O}}^{\text{rest}}(\mathbf{k})$ only contributes in non-local theory: $\mathrm{QED_L}$ at order $1/L^3$
- Leptonic decays problem: $g_{\mathcal{O}}^{\text{rest}}(\mathbf{k})$ unknown

General expansion of $g_{\mathcal{O}}^{\rm poles}(\mathbf{k})$

$$g_{\mathcal{O}}^{\rm poles}(\mathbf{k}) = \int_{\rm poles} \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, \mathbf{k}\right)}{k^2\left[(p-k)^2+m^2\right]}$$

- p on-shell: $p = (i\omega(\mathbf{p}), \mathbf{p})$, $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$
- Velocity $\mathbf{v} = \mathbf{p}/\omega(\mathbf{p})$
- Then: $f_{\mathcal{O}}\left(k_0, \mathbf{k}\right) = f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{v}\cdot\hat{\mathbf{k}}\right)$
- Two poles in the integrand (upper plane)

$$k_0 = i|\mathbf{k}|\,, \qquad k_0 = p_0 + i\omega(\mathbf{p}-\mathbf{k})$$

$$g_{\mathcal{O}}^{\rm poles}(\mathbf{k}) = i\,\frac{f_{\mathcal{O}}\left(i|\mathbf{k}|, |\mathbf{k}|, \mathbf{v}\cdot\hat{\mathbf{k}}\right)}{2|\mathbf{k}|\;\left[(i\omega(\mathbf{p}) - i|\mathbf{k}|)^2 + \omega(\mathbf{p}-\mathbf{k})^2\right]} + i\,\frac{f_{\mathcal{O}}\left(i\omega(\mathbf{p}) + i\omega(\mathbf{p}-\mathbf{k}), |\mathbf{k}|, \mathbf{v}\cdot\hat{\mathbf{k}}\right)}{2\omega(\mathbf{p}-\mathbf{k})\;\left[(i\omega(\mathbf{p}) + i\omega(\mathbf{p}-\mathbf{k}))^2 + |\mathbf{k}|^2\right]}$$

General expansion of $g_{\mathcal{O}}^{\mathrm{poles}}(\mathbf{k})$

- Let us expand the denominators for $\mathbf{k} = 2\pi\mathbf{n}/L$:

$$\frac{i}{L^3}\frac{f_{\mathcal{O}}\left(i|\mathbf{k}|, |\mathbf{k}|, \mathbf{v}\cdot\hat{\mathbf{k}}\right)}{2|\mathbf{k}|\ [(i\omega(\mathbf{p}) - i|\mathbf{k}|)^2 + \omega(\mathbf{p}-\mathbf{k})^2]} = \frac{1}{L}\frac{f_{\mathcal{O}}\left(i\frac{2\pi|\mathbf{n}|}{L}, \frac{2\pi|\mathbf{n}|}{L}, \mathbf{v}\cdot\hat{\mathbf{n}}\right)}{16\pi^2|\mathbf{n}|^2\ (1-\mathbf{v}\cdot\hat{\mathbf{n}})\omega(\mathbf{p})}$$

- Physical singularities: $1/|\mathbf{n}|$ (soft) and $1/(1-\mathbf{v}\cdot\hat{\mathbf{n}})$ (collinear)
- NB: Need to expand numerators too: Possible cancellations!
- Finite-size effects given by

$$\Delta_{\mathrm{poles}}\mathcal{O} = \left(\sum_{\mathbf{n}\in\Pi_{\mathbf{n}}} - \int d^3\mathbf{n}\right)\frac{1}{L^3}\, g_{\mathcal{O}}^{\mathrm{poles}}\left(\frac{2\pi\mathbf{n}}{L}\right)$$

- Define prescription-dependent coefficients

$$c_j(\mathbf{v}) = \left(\sum_{\mathbf{n}\in\Pi_{\mathbf{n}}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^j\,(1-\mathbf{v}\cdot\hat{\mathbf{n}})}$$

- $\mathrm{QED_L}$: $\Pi_{\mathbf{n}} = \mathbb{Z}^3 \setminus \{\mathbf{n}=\mathbf{0}\}$

Finite-size effects of $g_{\mathcal{O}}^{\mathrm{poles}}(\mathbf{k})$

$$\Delta_{\mathrm{poles}}\mathcal{O} = \frac{f_{\mathcal{O}}(0,0,0)\, c_2(\mathbf{v})}{16\pi^2 L\omega(\mathbf{p})}$$

$$+ \frac{f_{\mathcal{O}}^{(0,0,1)}(0,0,0)\omega(\mathbf{p})\left(c_1(\mathbf{v}) - c_1\right) + \left(f_{\mathcal{O}}^{(0,1,0)}(0,0,0) + i f_{\mathcal{O}}^{(1,0,0)}(0,0,0)\right) c_1(\mathbf{v})}{8\pi L^2 \omega(\mathbf{p})}$$

$$\frac{1}{8L^3}\left\{ \frac{c_0(\mathbf{v})}{\omega(\mathbf{p})}\left[f_{\mathcal{O}}^{(0,0,2)}(0,0,0)\omega(\mathbf{p})^2 + 2\left(f_{\mathcal{O}}^{(0,1,1)}(0,0,0) + i f_{\mathcal{O}}^{(1,0,1)}(0,0,0)\right)\omega(\mathbf{p}) \right.\right.$$

$$\left. + f_{\mathcal{O}}^{(0,2,0)}(0,0,0) + 2i f_{\mathcal{O}}^{(1,1,0)}(0,0,0) - f_{\mathcal{O}}^{(2,0,0)}(0,0,0) \right]$$

$$\left. - c_0\left[f_{\mathcal{O}}^{(0,0,2)}(0,0,0)\omega(\mathbf{p}) + 2 f_{\mathcal{O}}^{(0,1,1)}(0,0,0) + 2i f_{\mathcal{O}}^{(1,0,1)}(0,0,0) - \frac{f_{\mathcal{O}}(2i\omega(\mathbf{p}),0,0)}{\omega(\mathbf{p})^3} \right] \right\}$$

- $\mathrm{QED_L}$: $\mathbf{v} = |\mathbf{v}|(1,1,1)/\sqrt{3}$: (non-locality/collinear sing.)

$$c_0(|\mathbf{v}| = 0.27) = 1.03\,, \qquad c_0 = -1\,,$$
$$c_1(|\mathbf{v}| = 0.27) = 2.91\,, \qquad c_1 = 2.84,$$
$$c_0(|\mathbf{v}| = 0.91) = 2.12\,, \qquad c_j(|\mathbf{v}| \to 1) = \pm\infty$$

Finite-size effects of $\mathcal{O}(L)$

- The above approach extendable to any number of propagators/order
- In general we have

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3}\sum_{\mathbf{k}\in\Pi} - \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right)\left[g_{\mathcal{O}}^{\mathrm{poles}}(\mathbf{k}) + g_{\mathcal{O}}^{\mathrm{rest}}(\mathbf{k})\right]$$
$$= \Delta_{\mathrm{poles}}\mathcal{O} + \Delta_{\mathrm{rest}}\mathcal{O}$$

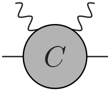
- If we know $g_{\mathcal{O}}^{\mathrm{poles}}(\mathbf{k})$ we get $\Delta_{\mathrm{poles}}\mathcal{O}$
- Caveat when we do not know $g_{\mathcal{O}}^{\mathrm{rest}}(\mathbf{k})$
- $g_{\mathcal{O}}^{\mathrm{rest}}(\mathbf{k})$ depends on skeleton expansion

Skeleton expansion

- Need to define kernels: Compton scattering amplitude

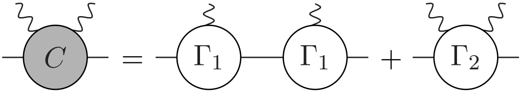
  [Di Carlo, Hansen, H.-T., Portelli 2022]

$$C_{\mu\nu}(p,k,q) =$$



$$\lim_{p^2 \to -m_{P,0}^2} C_{\mu\nu}(p,k,-k) = e^2 \int \mathrm{d}^4x \, e^{-ik\cdot x} \, \langle P, \mathbf{p} | \, T\left\{ J_\mu(x) J_\nu(0) \right\} | P, \mathbf{p} \rangle$$

- Method: Decompose into irreducible vertex functions $\Gamma_1 = \Gamma_\mu$, $\Gamma_2 = \Gamma_{\mu\nu}$



- Amplitude $C_{\mu\nu}(p,k,q)$ satisfies Ward identities:
  - Γ_μ and $\Gamma_{\mu\nu}$ must satisfy them as well, but arbitrary separation!
  - Non-physical (off-shell) and physical (structure-dependent) terms

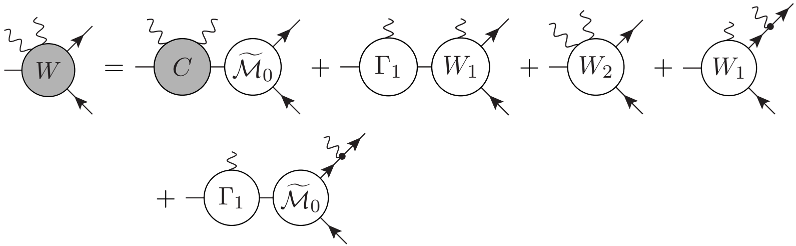
Self-energy in the moving frame

- We can choose the moving frame with $\mathbf{p} \neq \mathbf{0}$: New result

$$\Delta m_P^2(L) = e^2 m_P^2 \Bigg\{ \frac{1}{\gamma(|\mathbf{v}|)} \frac{c_2(\mathbf{v})}{4\pi^2\, m_P L} + \frac{c_1}{2\pi(m_P L)^2} + \frac{c_0}{(m_P L)^3}\left[\frac{\left(\gamma(|\mathbf{v}|)^2 - 1\right)\left(1 - \frac{2\langle r_P^2\rangle m_P^2\, \gamma(|\mathbf{v}|)^2}{3}\right)^2}{2\gamma(|\mathbf{v}|)^3} - \frac{2\langle r_P^2\rangle m_P^2}{3}\gamma(|\mathbf{v}|)\right] + \frac{\widetilde{\mathcal{C}}}{(m_P L)^3} + \mathcal{O}\left[\frac{1}{(m_P L)^4}, e^{-m_P L}\right]\Bigg\}$$

- Lorentz contraction factor: $\gamma(|\mathbf{v}|) = (1 - |\mathbf{v}|^2)^{-1/2}$
- Point-like approximation agrees with [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
- Branch-cut: Specific to $\mathrm{QED_L}$ (not in $\mathrm{QED_C}$ [Lucini, Patella, Ramos, Tantalo 2016])
- Only the velocity-dependent coefficient $c_2(\mathbf{v})$ appears

Leptonic decays $P \to \ell\nu$



- Finite-size effects of virtual decay rate $\Gamma(P \to \ell\nu)$: $Y^{(n)}(L)$
- W_1: $A(k^2,(p+k)^2)$, $V(k^2,(p+k)^2)$, $H_{1,2}(k^2,(p+k)^2)$: appear in $P^- \to \ell^-\nu_\ell\gamma^{(*)}$
- W_2: Three additional form factors $C_{1,2,3}(k^2,(p+k)^2)$ from $P^- \to \ell^-\nu_\ell\gamma^{(*)}\gamma^{(*)}$
- Skeleton expansion: $W_{1,2}$ depend on off-shellness that has to cancel
- Does not occur due to $g_{\mathcal{O}}^{\mathrm{rest}}(\mathbf{k})$ $\Longrightarrow$ On-shell representation

Leptonic decays: Finite-size effects

$$Y^{(3)}(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3 - 2\,c_3(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\log\left(\frac{m_W L}{4\pi}\right)$$

$$- 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi\,[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{4\pi^2}{3\,f_P(1-r_\ell^2)(1+r_\ell^2)^3\,(m_P L)^3}\left\{24\,m_P\,(1+r_\ell^2)^2\,c_0(\mathbf{v}_\ell)\,\left[F_V^P - F_A^P + 2m_P^2 r_\ell^2\,A^{(0,1)}\left(0,-m_P^2\right)\right]\right.$$

$$c_0\left[f_P\,(-1+r_\ell^2)\,\left(\left\langle r_P^2\right\rangle m_P^2(1+r_\ell^2)^3 - 24\,(2+r_\ell^2)\right)\right.$$

$$+ 6\,m_P(1+r_\ell^2)\left[F_A^P\,(1+r_\ell^2)^3 + 4\,A\left(-m_P^2(1+r_\ell^2)^2, -m_P^2\,r_\ell^4\right) r_\ell^2\left(-2+r_\ell^2+r_\ell^4\right)\right.$$

$$- 2\,(1+r_\ell^2)\left((1+r_\ell^2)\,F_V^P - m_P(-1+r_\ell^4)\,H_1\left(0,-m_P^2\right)\right.$$

$$- 2\,m_P\,(-2+r_\ell^2+r_\ell^4)\,H_1\left(-m_P^2(1+r_\ell^2)^2, -m_P^2\,r_\ell^4\right)$$

$$\left.\left.\left.\left. - m_P^3(-1+r_\ell^4)\,H_2\left(0,-m_P^2\right) + m_P^2(1+r_\ell^2)^2\,A^{(0,1)}\left(0,-m_P^2\right)\right)\right]\right]\right\} + \frac{\mathcal{C}_\ell}{(m_P L)^3}$$

Leptonic decays: Finite-size effects

- Structure-dependent finite-size effects through order $1/L^2 +$ point-like $1/L^3$ in [Di Carlo, Hansen, H.-T., Portelli 2022]
- New: Structure-dependent $1/L^3$ part: F_A^P, F_V^P, $A\left(-m_P^2(1+r_\ell^2)^2, -m_P^2\, r_\ell^4\right)$, $A^{(0,1)}\left(0, -m_P^2\right)$, $H_{1,2}\left(0, -m_P^2\right)$, $H_1\left(-m_P^2(1+r_\ell^2)^2, -m_P^2\, r_\ell^4\right)$
- Obtainable from chiral perturbation theory [Bijnens, Ecker, Gasser 1992], lattice [RM-123/Soton 2020/2022, RBC/UKQCD 2023], experiment [...]
- The branch-cut $\mathcal{C}_\ell$ still unknown so no prediction
- Important: Collinear divergence contributes with $c_0(\mathbf{v}_\ell)$ at order $1/L^3$
- New finding but expected since collinear singularities physical

Conclusions

- Finite-size effects at order $1/L^3$ difficult in $\mathrm{QED_L}$: Branch-cut
- Branch-cut depends on the effective representation of vertices
- New: $1/L^3$ excluding branch-cut
  Self-energy in moving frame and leptonic decays
- Collinear divergences $c_j(\mathbf{v})$: Appear from underlying physics
- Life would be simpler without the $1/L^3$: On-going work in Edinburgh (Lattice 23)

Back-up slides

Decomposing vertex functions

- Step 2: Form-factor decomposition (structure-dependence!)

$$\Gamma_\mu(p,k) = (2p+k)_\mu\, F(k^2,(p+k)^2,p^2) + k_\mu\, G(k^2,(p+k)^2,p^2)$$

- Contains both on-shell and off-shell dependence

$$F^{(1,0,0)}(0,-m_P^2,-m_P^2) \equiv F'(0) = -\langle r_P^2\rangle/6$$

- $F^{(0,0,n)}(0,-m_P^2,-m_P^2)$: Unphysical derivative! $\longrightarrow$ Must always cancel in the end!
- How must they cancel, and what about $G(k^2,(p+k)^2,p^2)$?

Decomposing vertex functions

- Step 3: Use Ward identities, e.g.

$$k_\mu \Gamma^\mu(p,k) = D(p+k)^{-1} - D(p)^{-1}$$

- Define full propagator ($Z(p^2)$: z_n [BMW 2015; RM-123/Soton 2017])

$$D(p) = \frac{Z(p^2)}{p^2 + m_P^2}$$

- Ward identity yields G as a function of F and

$$F(0, p^2, -m_P^2) = F(0, -m_P^2, p^2) = Z(p^2)^{-1}$$

- Example relation: $z_1 = F^{(0,0,1)}(0, -m_P^2, -m_P^2)$
- Unphysical derivative! $\longrightarrow$ Must always cancel in the end!
- Equivalently: We could put all non-physical quantities to zero directly

$$F(k^2, (p+k)^2, p^2) \to F(k^2) = 1 + k^2 F'(0) + \ldots$$
$$Z(p^2) \to 1$$

- Off-shellness does not cancel at order $1/L^3$ for self-energy or leptonic decays $\implies$ $\Delta_{\text{rest}}\mathcal{O}$ depends on definition of vertex functions