Lattice approaches to microphysical bubble dynamics

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Cosmological 1st-order phase transitions

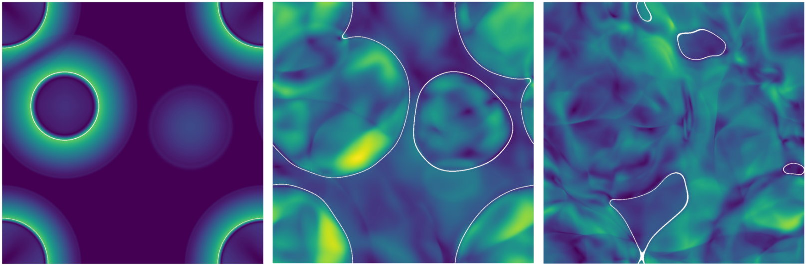
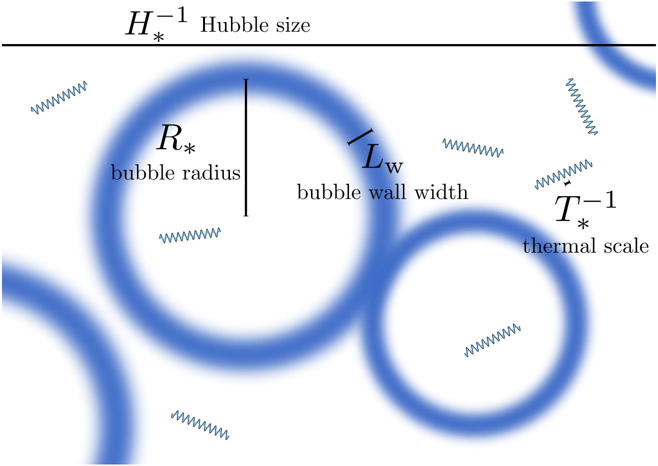


Figure: Cutting et al. arXiv:1906.00480.

- Bubbles nucleate, expand and collide
- This creates long-lived fluid flows
- And creates gravitational waves: $\Box h_{ij}^{(\mathrm{TT})} \sim T_{ij}^{(\mathrm{TT})}$

Scales



$$\underbrace{H_*^{-1} \gg R_*}_{\text{fluid}} \gg \underbrace{L_\mathrm{w} \gg T^{-1}}_{\text{microphysical}}$$

Theoretical uncertainties

GW signal depends strongly on 4 phase transition quantities,

$$\Omega_{\rm GW} = F(T_*, R_*, \alpha_*, v_{\rm w}),$$

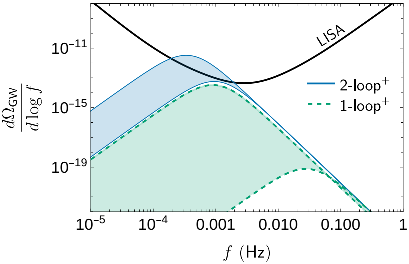
T_* : transition temperature,

R_* : bubble radius,

α_* : transition strength,

$v_{\rm w}$: bubble wall speed.

Each involves real-time physics.



Large uncertainties linked to microphysical predictions.

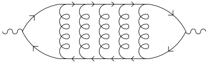
OG & Tenkanen 2104.04399

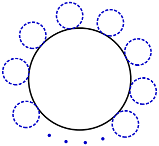
A direct attack is difficult

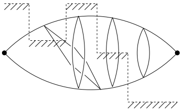
At high temperatures, time-dependent n-point functions,

$$(-\Box + V'')G(x,x') = \delta(x-x') + \int_y \Pi(x,y)G(y,x'),$$

do not admit a simple perturbative expansion.







Arnold, Moore & Yaffe '03, Kapusta '79, Jeon '94

Infrared strong coupling

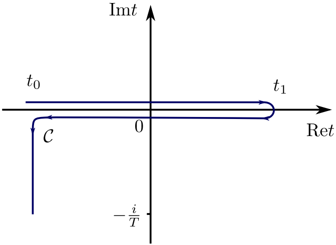
Infrared bosons are highly occupied; the effective expansion parameter α_{eff} grows

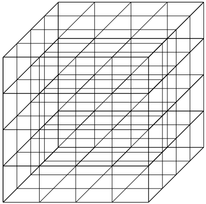
$$\alpha_{\text{eff}} \sim g^2 \frac{1}{e^{E/T} - 1} \approx g^2 \frac{T}{E}$$

Softer modes are classically occupied and more strongly coupled:

$$\begin{aligned} \text{hard}: \quad & E \sim T \Rightarrow \alpha_{\text{eff}} \sim g^2 \sim 0.03, \\ \text{soft}: \quad & E \sim gT \Rightarrow \alpha_{\text{eff}} \sim g \sim 0.18, \\ \text{ultrasoft}: \quad & E \sim g^2 T \Rightarrow \alpha_{\text{eff}} \sim g^0 \sim 1. \end{aligned}$$

The lattice is limited





- Real-time sign problem thwarts importance sampling,

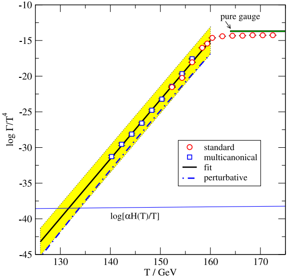
$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle_{\mathsf{qm}} = \frac{1}{Z}\int_{\mathcal{C}} \mathcal{D}\phi \; \mathcal{O}(t)\mathcal{O}(0)e^{iS[\phi]}.$$

- Chiral fermions, like e_{L}, e_{R}, can't be simulated on a lattice.

Nielsen & Ninomiya '81

Approach

Inspired by e.g. lattice results for the electroweak sphaleron rate,



d'Onofrio & Rummukainen '14
Moore '98, '00

See Jaakko Annala's talk next

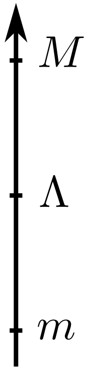
We'll discuss **effective field theory** approaches.

Real-time effective theories

Euclidean EFT

Consider a simple model with a scale hierarchy $m \ll M$,

$$\begin{aligned}\mathscr{L} =& \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}(\partial_\mu \chi)^2 + \frac{M^2}{2}\chi^2 + \frac{\kappa}{4!}\chi^4 \\ &+ \frac{g^2}{4}\phi^2\chi^2.\end{aligned}$$

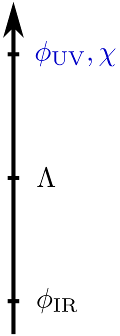


Wilsonian EFT

- Split degrees of freedom $\{\phi, \chi\}$ based on energy.

- Integrate out the UV modes:

$$\int \mathcal{D}\phi \int \mathcal{D}\chi \; e^{-S[\phi,\chi]} = \int \mathcal{D}\phi_{\text{IR}} \left(\int \mathcal{D}\phi_{\text{UV}} \int \mathcal{D}\chi \; e^{-S[\phi,\chi]} \right)$$
$$= \int \mathcal{D}\phi_{\text{IR}} \; e^{-S_{\text{eff}}[\phi_{\text{IR}}]}$$



- Expansion in p^2/Λ^2 and m^2/Λ^2 becomes an expansion in p^2/M^2 and m^2/M^2

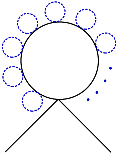
Burgess '21, Hirvonen '22

Resummations with EFT

By first integrating out the UV modes

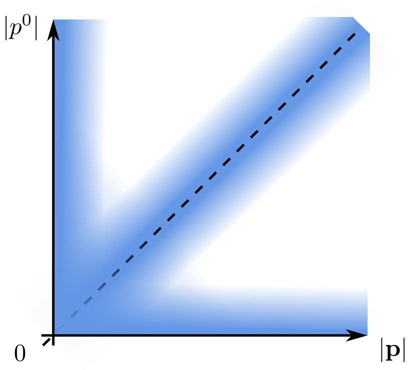
$$
\begin{aligned}
S_{\text{eff}}[\phi_{\text{IR}}] &= S_\phi[\phi_{\text{IR}}] - \log \int \mathcal{D}\phi_{\text{UV}} \mathcal{D}\chi \; e^{-S[\phi_{\text{IR}}+\phi_{\text{UV}},\chi]+S_\phi[\phi_{\text{IR}}]}, \\
&\approx S_\phi[\phi_{\text{IR}}] + \int_x \frac{1}{2}(m_{\text{eff}}^2 - m^2)\phi_{\text{IR}}^2,
\end{aligned}
$$

the daisy resummations arise naturally.



So do all other necessary resummations, order by order.

Scale hierarchies in real time



More possible scale hierarchies:

- $|p^0|, |\mathbf{p}| \ll \Lambda$
- $|\mathbf{p}| \ll \Lambda$, $|p^0| \sim \Lambda$
- $|p^0| \ll \Lambda$, $|\mathbf{p}| \sim \Lambda$
- $||p^0| - |\mathbf{p}|| \ll \Lambda$, $|p^0|, |\mathbf{p}| \sim \Lambda$

Real-time Wilsonian effective actions

Consider our Euclidean effective action from earlier,

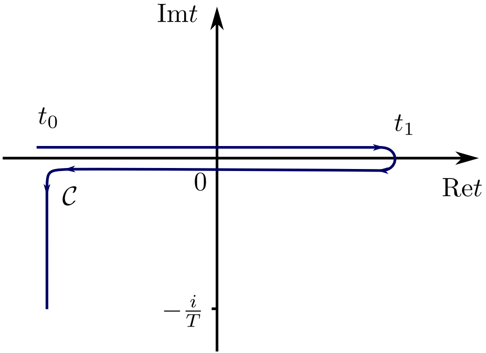
$$S_{\text{eff}}[\phi_{\text{IR}}] = S_\phi[\phi_{\text{IR}}] - \log \int \mathcal{D}\phi_{\text{UV}}\mathcal{D}\chi \; e^{-S[\phi_{\text{IR}}+\phi_{\text{UV}},\chi]+S_\phi[\phi_{\text{IR}}]}$$

How can we generalise this to real-time?

- consider soft external modes: $|p^0|, |\mathbf{p}| \sim gT$
- Integrate over hard internal loops $k_n, |\mathbf{k}| \sim \pi T$
- Taylor expand final result in soft quantities.

see e.g. Laine & Vuorinen '17

Quantum thermal evolution



$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle_{\mathsf{qm}} = \frac{1}{Z}\mathrm{Tr}\left[e^{-\hat{H}/T}\left(e^{i\hat{H}t}\mathcal{O}(0)e^{-i\hat{H}t}\right)\mathcal{O}(0)\right]$$
$$= \int_{\mathcal{C}} \mathcal{D}\phi\,\mathcal{O}(t)\mathcal{O}(0)e^{iS[\phi]}$$

Classicalisation

- Bose enhancement of IR modes

$$n_{\mathrm{B}}(E) = \frac{1}{e^{E/T} - 1}, \\ \approx \frac{T}{E} \gg 1.$$

- Dynamics of QFT at nucleation scale ($\Lambda_{\mathrm{nucl}} \ll T$) expected to be quasi-classical.

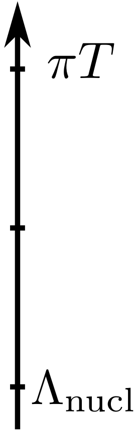


Figure: Nucleation scale much lower than thermal scale.

Classical thermal evolution

Hamilton's equations determine real-time correlation functions,

$$\dot{\phi}(t,x) = \{\phi(t,x), H\},$$
$$\dot{\pi}(t,x) = \{\pi(t,x), H\},$$

with thermal initial conditions,

$$\langle \phi(0,x_1)\phi(0,x_2)\rangle_{\text{cl}} \equiv \frac{1}{Z_{\text{cl}}}\int \mathcal{D}\phi\mathcal{D}\pi\,\phi(0,x_1)\phi(0,x_2)e^{-H[\phi,\pi]/T}.$$

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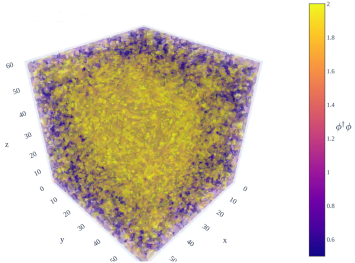
Looks pretty far from the quantum case

Quantum versus classical

- Classical UV catastrophe - the cut-off scale dominates!



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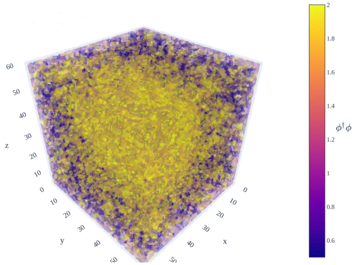

- Dimensional reduction counterterms cancel UV catastrophe $\rightarrow$ finite result (c.f. Kari Rummukainen's talk). Aarts & Smit '97

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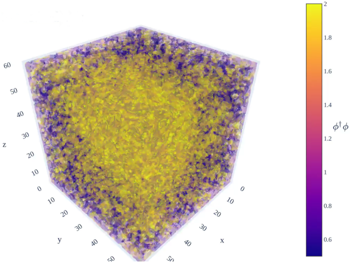
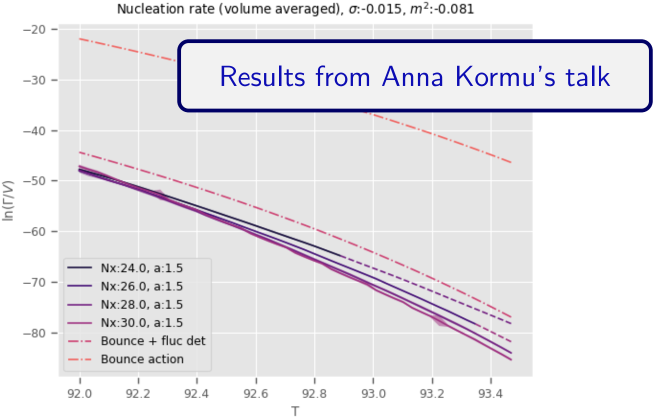


- Dimensional reduction counterterms cancel UV catastrophe $\rightarrow$ finite result (c.f. Kari Rummukainen's talk). Aarts & Smit '97
- Moreover, the finite classical and quantum remainders agree!

$$\langle\phi(t_1,\mathbf{p}_1)\phi(t_2,\mathbf{p}_2)\rangle_{\mathrm{qm}} \approx \underbrace{\langle\{\phi(t_1,\mathbf{p}_1),\phi(t_2,\mathbf{p}_2)\}\rangle_{\mathrm{cl}}}_{\text{no sign problem}}$$

Bödeker '97

Effective stochastic theory on the lattice



$$\frac{\Gamma}{V} \sim A e^{-B}$$

$$H_{\rm eff} = \int \mathrm{d}^3 x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_i \phi)^2 + \sigma_{\rm eff}\phi + \frac{1}{2}(m_{\rm eff}^2 + \delta m_{\rm eff}^2)\phi^2 + \frac{g_{\rm eff}^2}{4!}\phi^4 \right]$$

OG, Kormu & Weir (forthcoming), Moore, Rummukainen & Tranberg '01

Soft gauge fields

The above strategy is more complicated for gauge fields:

- Thermal initial conditions are as before, based on the 3d EFT.
- Evolution equations (here for Abelian-Higgs model) become

$$D_\mu D^\mu \phi = -m_{\text{eff}}^2 \phi - 2\lambda(\phi^* \phi)\phi,$$
$$\partial_\mu F^{\mu\nu} = \Pi^{\mu\nu} A_\mu + 2ie(\phi^* D^\nu \phi - \phi D^\nu \phi^*),$$

where

$$\Pi^{\mu\nu}(P) = -\frac{e^2 T^2}{3} \int \frac{d\Omega_v}{4\pi} \left[n^\mu n^\nu + v^\mu v^\nu \frac{p_0}{v \cdot P} \right].$$

- Nonlocality can be resolved by adding new fields.

Hu & Miller '96, Iancu '98

Bödeker, Moore & Rummukainen '99

Recent developments in hard-thermal loops

The hard-thermal loop effective theory for gauge fields (here shown for QED)

$$\partial_\mu F^{\mu\nu} = (\Pi^{\mu\nu}_{\mathrm{LO}} + \Pi^{\mu\nu}_{\mathrm{NLO}})A_\mu,$$

has been recently extended to NLO,

$$\Pi^{\mu\nu}_{\mathrm{NLO}}(P) = -\frac{e^4 T^2}{8\pi^2}\int \frac{d\Omega_v}{4\pi}\Big\{ v^\mu v^\nu \left[\frac{(p^0)^2}{(v\cdot P)^2} - \frac{2p^0}{v\cdot P}\right] + [v^\mu n^\nu + n^\mu v^\nu]\,\frac{p^0}{v\cdot P} - n^\mu n^\nu \Big\}.$$

Carignano et al. '20, Ekstedt '23, Gorda et al. '23

Langevin equations

Influence functional

Split the field based on spatial momentum

$$\Phi(t,\mathbf{p}) = \underbrace{\theta(\Lambda - |\mathbf{p}|)\Phi(t,\mathbf{p})}_{\Phi_{\text{IR}}} + \underbrace{\theta(|\mathbf{p}| - \Lambda)\Phi(t,\mathbf{p})}_{\Phi_{\text{UV}}},$$

Bödeker, McLerran & Smilga '95, Lombardo & Mazzitelli '95

and integrate over the UV modes,

$$\int \mathcal{D}\Phi\mathcal{D}\Phi' \rho_i e^{i(S[\Phi]-S[\Phi'])}$$
$$= \int \mathcal{D}\Phi_{\text{IR}}\mathcal{D}\Phi'_{\text{IR}}\rho_{\text{IR},i} e^{i(S[\Phi_{\text{IR}}]-S[\Phi'_{\text{IR}}]+S_{\text{IF}}[\Phi_{\text{IR}},\Phi'_{\text{IR}}])}.$$

The influence functional S_{IF} gives the effect of the UV modes, in the in-in formalism.

Feynman & Vernon '63

Complex influence functionals

In general the evolution of the IR modes is nonunitary

$$e^{i(S[\Phi_{IR}]-S[\Phi'_{IR}]+\mathrm{Re}S_{IF}[\Phi_{IR},\Phi'_{IR}])} \times e^{-\mathrm{Im}S_{IF}[\Phi_{IR},\Phi'_{IR}]}.$$

This complicates the naive semiclassical limit,

$$\frac{\delta}{\delta\Phi_{IR}}(S[\Phi_{IR}] + \mathrm{Re}S_{IF}[\Phi_{IR},\Phi'_{IR}] + i\mathrm{Im}S_{IF}[\Phi_{IR},\Phi'_{IR}]) = 0 \text{ ???}$$

Stochastic semiclassical limit

A possible solution is to introduce new stochastic variables, e.g.

$$
\begin{aligned}
e^{-\mathrm{Im}S_{\mathrm{IF}}[\Phi_{\mathrm{IR}},\Phi'_{\mathrm{IR}}]} &\to e^{-\frac{1}{2}\Phi_{\mathrm{IR}}\cdot\mathcal{I}_2\cdot\Phi_{\mathrm{IR}}}, \\
&= \frac{1}{\sqrt{\det\mathcal{I}}}\int\mathcal{D}\chi e^{-\frac{1}{2}\chi\cdot\mathcal{I}_2^{-1}\cdot\chi}e^{i\chi\cdot\Phi_{\mathrm{IR}}}.
\end{aligned}
$$

The effective action for Φ_{IR} is then real.

The semiclassical equations of motion become Langevin,

$$
\begin{aligned}
\frac{\delta}{\delta\Phi_{\mathrm{IR}}}S[\Phi_{\mathrm{IR}}] + \frac{\delta}{\delta\Phi_{\mathrm{IR}}}\mathrm{Re}S_{\mathrm{IF}}[\Phi_{\mathrm{IR}},\Phi'_{\mathrm{IR}}] &= \chi, \\
\langle\chi(x)\chi(y)\rangle &= \mathcal{I}_2(x,y).
\end{aligned}
$$

Effective stochastic $\lambda\phi^4$

Explicitly, for the $\lambda\phi^4$ theory,

$$-\Box\phi_{\rm IR}(x) + m^2\phi_{\rm IR}(x) + \lambda\phi_{\rm IR}(x)^3 + \int_{t_i}^{t} d^4y \,{\rm Re}\Gamma^{(2)}(x-y)\phi_{\rm IR}(y) = \chi(x) + \ldots$$

where the stochastic variable satisfies

$$\langle\chi(x)\chi(y)\rangle = {\rm Im}\Gamma^{(2)}(x-y),$$

and where $\Gamma^{(2)}$ is the UV contribution to the IR self-energy. Here we have made an expansion in powers of $\phi_{\rm IR}$.

Greiner & Müller '·97

Time evolution of ultrasoft gauge bosons

Starting from HTLs for gauge fields, one can integrate out the soft scale, to arrive at an effective theory for the ultrasoft scale.

soft: $E \sim gT$

ultrasoft: $E \sim g^2 T/\pi$

To leading-log order the result is first-order Langevin,

$$(D_t A_i)^a = -\gamma \frac{\delta S_3}{\delta A_i^a} + \xi_i^a$$

where S_3 is the Euclidean action of the 3d EFT, $\gamma \sim \log(1/g)/T$ is the colour damping, and ξ_i is a Gaussian noise satisfying

$$\langle \xi_i^a(t, \mathbf{x}) \xi_j^b(u, \mathbf{y}) \rangle = 2\gamma \delta_{ij} \delta^{ab} \delta(\mathbf{x} - \mathbf{y}) \delta(t - u).$$

Bodeker '98

Time evolution of gauge-Higgs system

For gauge-Higgs theory, the coupled Langevin equations read

$$(D_t A_i)^a = -\gamma \frac{\delta S_3}{\delta A_i^a} + \xi_i^a,$$

$$D_t \phi = -\eta\gamma \frac{\delta S_3}{\delta \phi^\dagger} + \xi_\phi$$

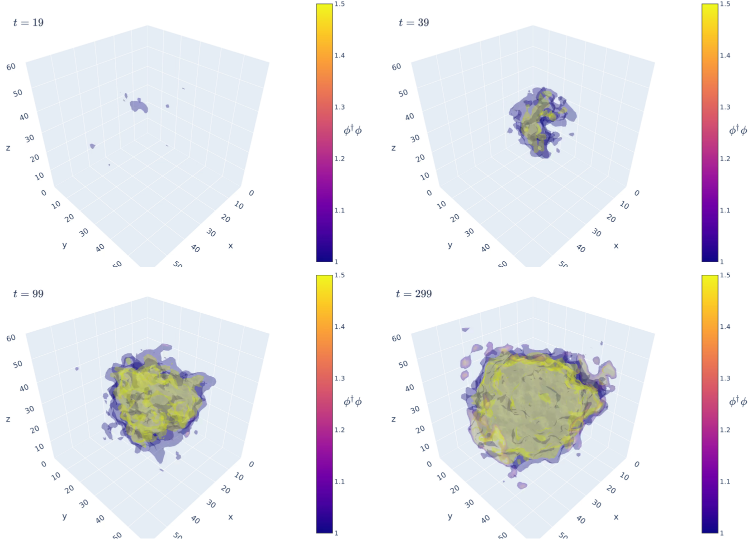
where the Higgs noise terms satisfies

$$\langle \xi_\phi(t, \mathbf{x}) \xi_\phi^\dagger(u, \mathbf{y}) \rangle = 2\eta\gamma \mathbb{1} \delta(\mathbf{x} - \mathbf{y}) \delta(t - u),$$

with $\eta \sim 1/g^2 \gg 1$, so that the Higgs evolves faster.

Bodeker '98, Moore '00

Early-time bubble dynamics



OG, Güyer & Rummukainen '22

Conclusions

- Lattice simulations are invaluable for phase transitions.
- Direct real-time lattice simulations not possible.
- Classical stochastic theories describe IR dynamics at high-T:
  - Initial fluctuations + classical evolution
  - Langevin stochastic evolution
- Successfully applied to bubble nucleation, and sphaleron rate.

  Anna Kormu & Jaakko Annala's talks
- Bubble wall speed? Electroweak baryogenesis? Initial studies, but not yet fully exploited. Mou, Saffin & Tranberg '20

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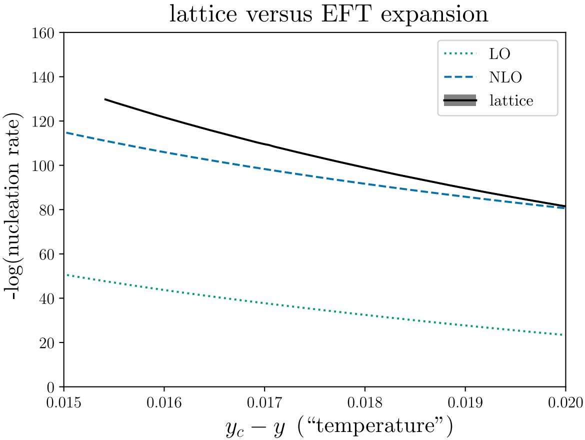
Thanks for listening!

Backup slides

Electroweak nucleation rate results



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