Entanglement entropy and lattice gauge theory

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➔ Quantum physical implementation of conservation laws

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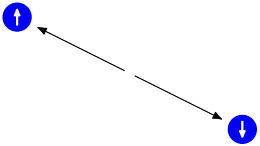

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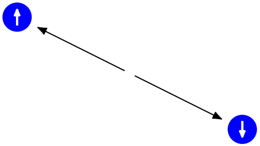
- Pair creation from vacuum: $s = 0 \longrightarrow s_1 + s_2 = 0$



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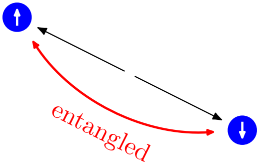

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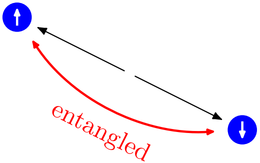

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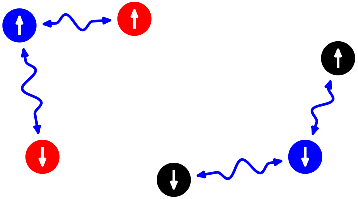

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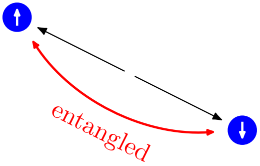
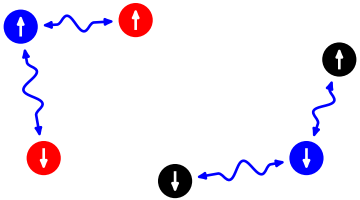


- In a quantum field theory:

  ➔ correlations



What is entanglement entropy?

- Preliminaries:

  Hilbert space: $\mathcal{H}$, state vector: $|\psi\rangle \in \mathcal{H}$

  Density matrix:

  $\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$, $|\psi_i\rangle \in \mathcal{H} \quad \forall\, i$, $\sum_i p_i = 1$

  $\mathrm{tr}(\rho) = 1$

  pure state: $\rho = |\psi\rangle\langle\psi|$

  ➔ $\rho^2 = \rho$ (projector) ➔ $\mathrm{tr}\big(\rho^2\big) = 1$

  mixed state: $\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$

  ➔ $\rho^2 \neq \rho$ (not projector) ➔ $\mathrm{tr}\big(\rho^2\big) < 1$

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What is entanglement entropy?

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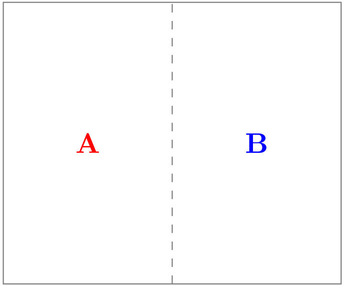
  pick pure state: $|\psi\rangle_{AB} \in \mathcal{H}_{AB}$

  pick orthonormal bases: $|n\rangle_A \in \mathcal{H}_A$, $|m\rangle_B \in \mathcal{H}_B$

  ➔ $|\psi\rangle_{AB} = \sum_{mn} a_{mn}\, |m\rangle_A \otimes |n\rangle_B \quad , \quad \sum_{mn} |a_{mn}|^2 = 1$

  ➔ $\rho_{AB} = |\psi\rangle_{AB}\langle\psi| = \sum_{mnkl} a_{mn} a^*_{kl} |m\rangle_A\langle k| \otimes |n\rangle_B\langle l|$

  (notation: $|\psi\rangle_C\langle\psi| = |\psi\rangle_C \otimes {}_C\langle\psi|$)



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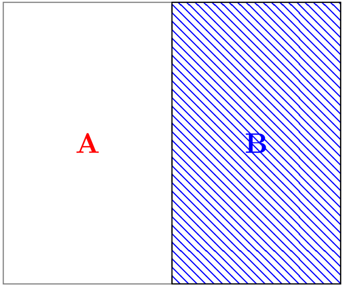

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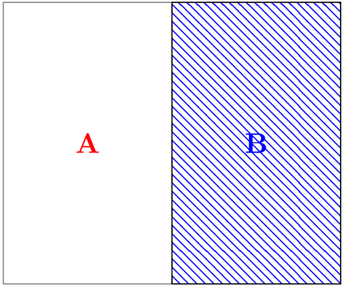

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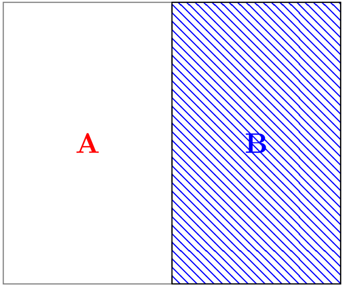
  ➔ $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$ $\Longrightarrow$ $\mathrm{tr}\left(\rho_A^2\right) = 1 \quad \Longrightarrow \quad$ no entanglement



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➔ Purity: $\mathrm{tr}\left(\rho_A^2\right)$

➔ Rényi entropies: $H_s(A) = -\frac{1}{s-1} \log \mathrm{tr}\left(\rho_A^s\right) \quad , \quad s = 2, 3, \ldots$

➔ Entanglement entropy: $S_{EE}(A) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}(\rho_A^s)}{\partial s} = \lim_{s\to 1} \frac{\partial((s-1)H_s(A))}{\partial s} = \lim_{s\to 1} H_s(A)$

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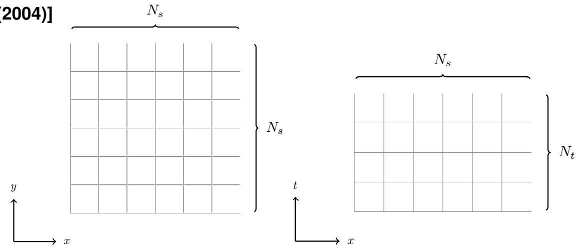
Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U]\ \mathrm{e}^{-S_G[U]}$



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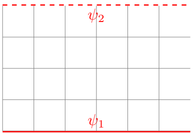

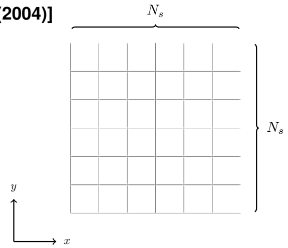

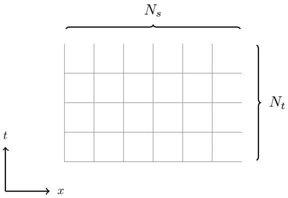

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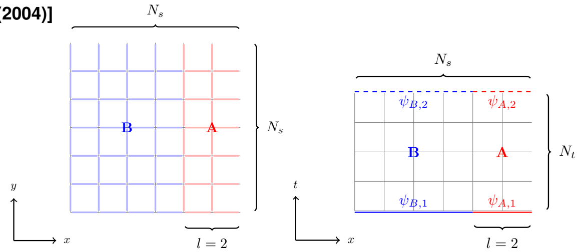

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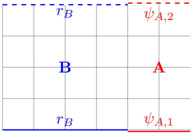

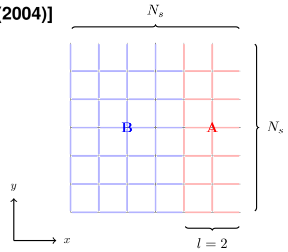

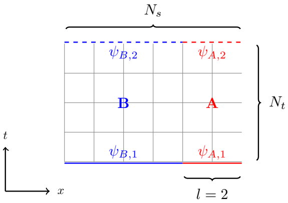

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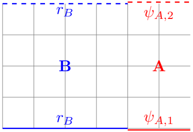
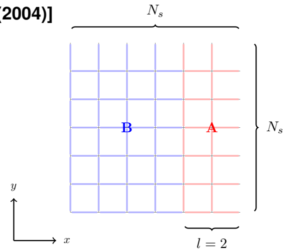
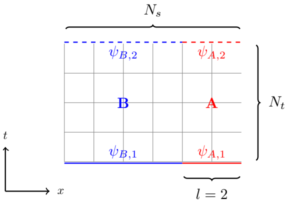
$\langle \psi_{A,1} | \rho_A | \psi_{A,2} \rangle =$ 

  ➔ Entanglement entropy:

  $S_{EE} = -\,\mathrm{tr}_A(\rho_A \log \rho_A)$ (how ?)





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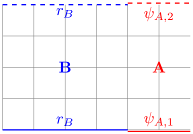
  $\langle \psi_{A,1} | \rho_A | \psi_{A,2} \rangle =$ 

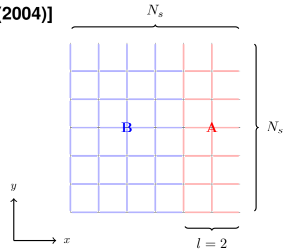
  ➔ Replica method for s-th Rényi entropy:

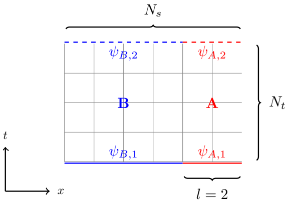
$$H_s(l, N_t, N_s) = \frac{1}{1-s} \log \mathrm{tr}(\rho_A^s) = \frac{1}{1-s} \log \frac{Z_c(l, s, N_t, N_s)}{Z^s(N_t, N_s)}$$

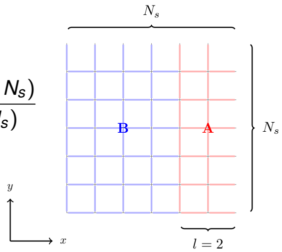
with "cut partition function" $Z_c(l, s, N_t, N_s)$

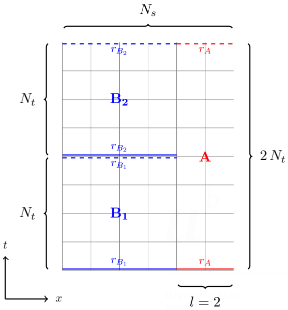
$\rightarrow \quad Z_c(l = 0, s, N_t, N_s) = Z^s(N_t, N_s) \quad \forall s \in \mathbb{N}$

$\rightarrow \quad Z_c(l = N_s, s, N_t, N_s) = Z(s\, N_t, N_s) \quad \forall s \in \mathbb{N}$









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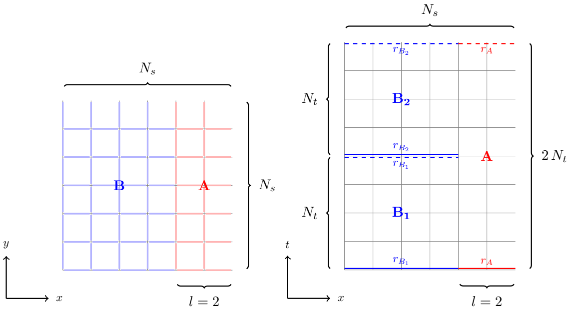
➔ measure free energy difference



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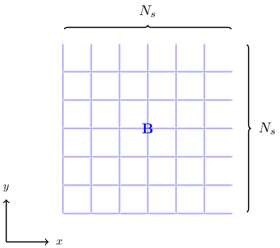

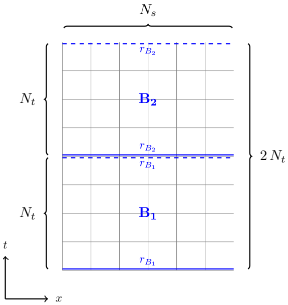

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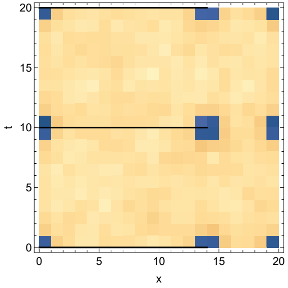
Issue: UV-divergent piece $\quad \frac{S_{EE}}{|\partial A|} = \boxed{\frac{C_0}{\epsilon^2}} - \frac{C}{l^q} + \text{(finite)}$



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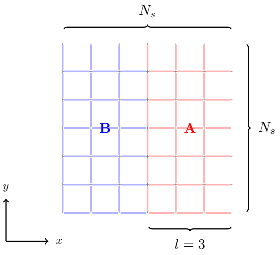

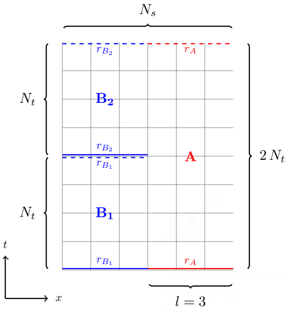

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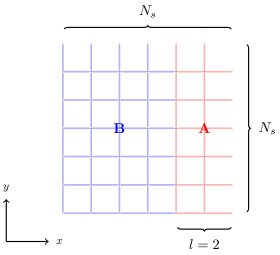

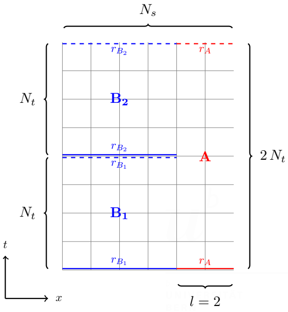

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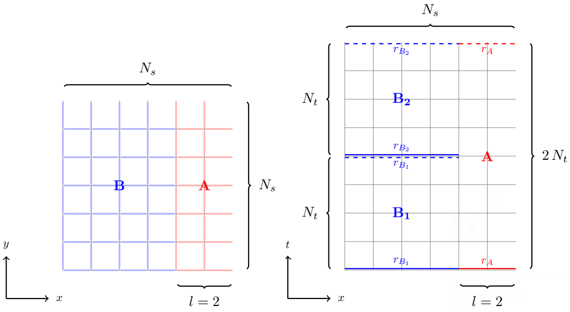

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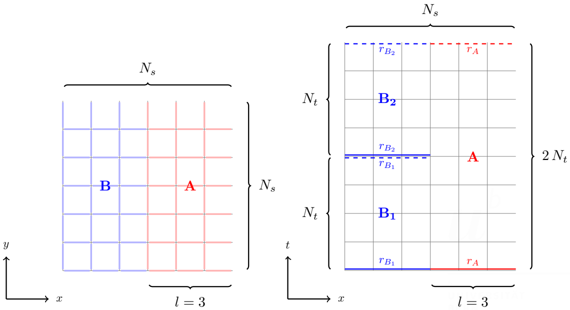

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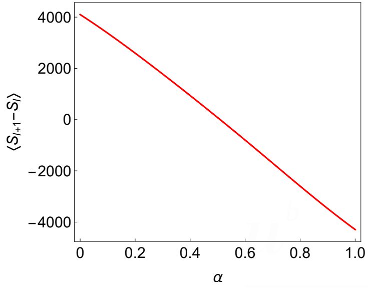

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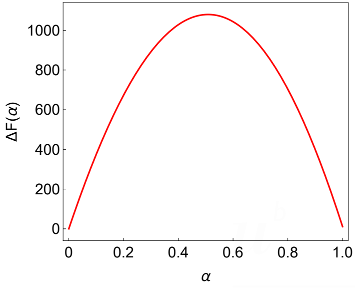

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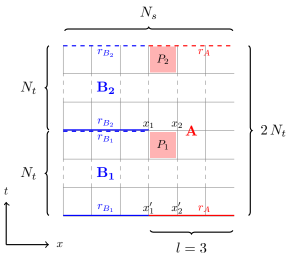
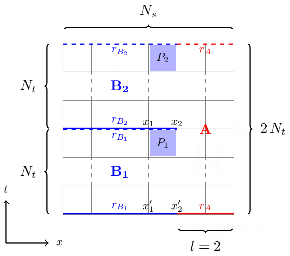
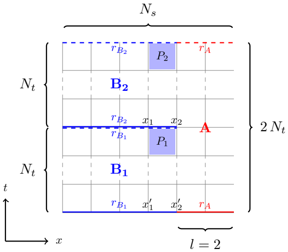
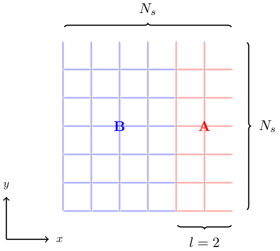




Entangling surface deformation method

How can we avoid (huge) free energy barriers?





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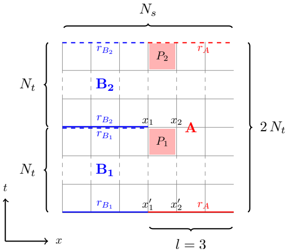

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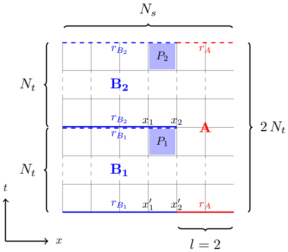

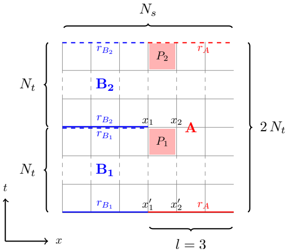

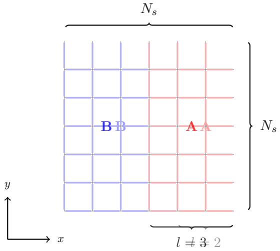

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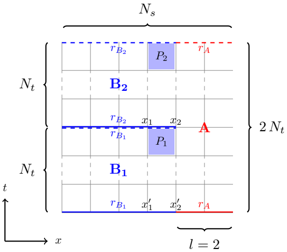

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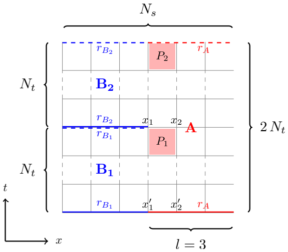

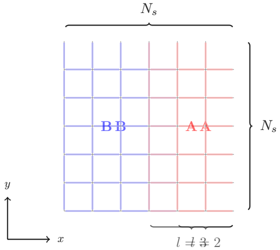

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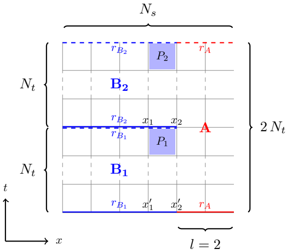

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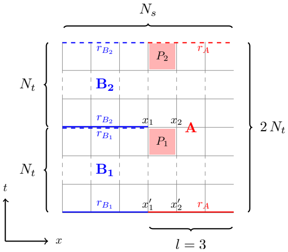

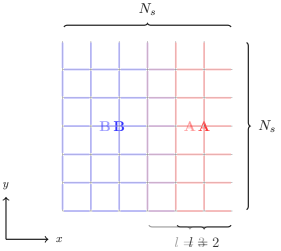

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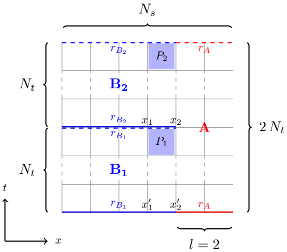

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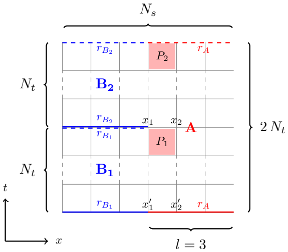

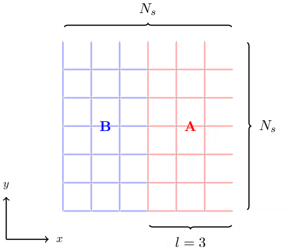

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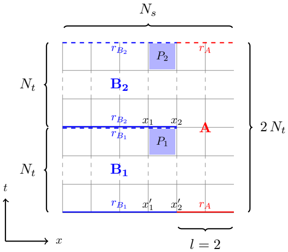

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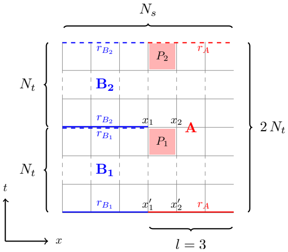

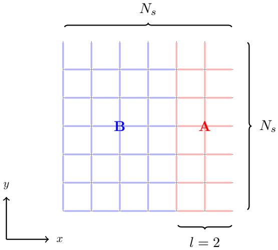

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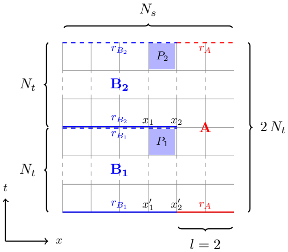

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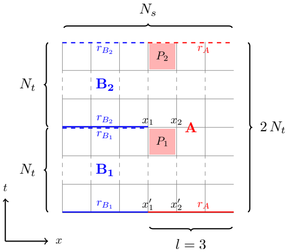

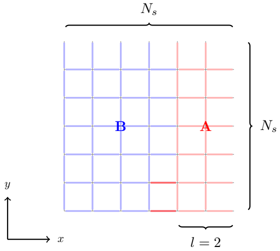

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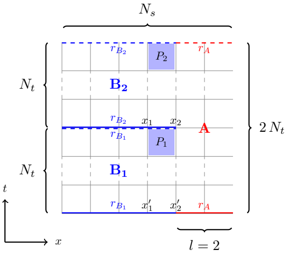

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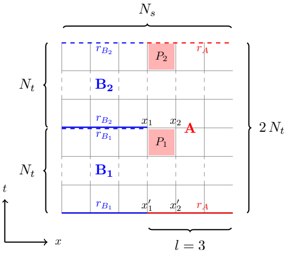

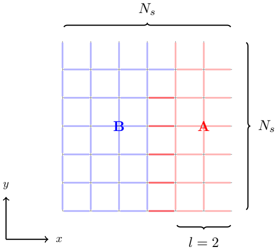

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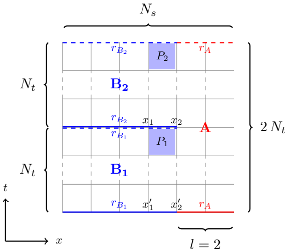

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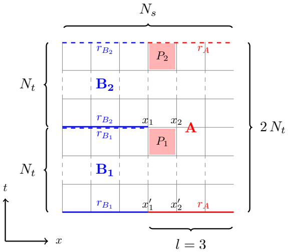

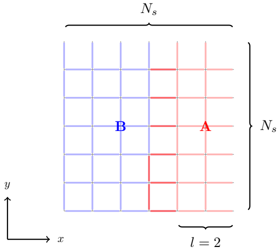

Entangling surface deformation method

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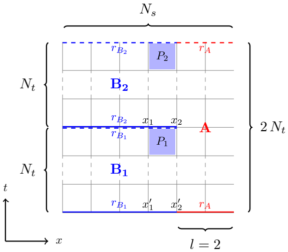

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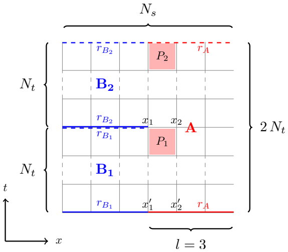

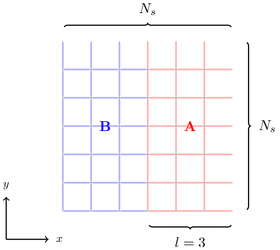

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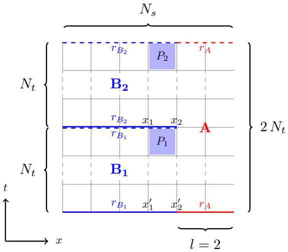

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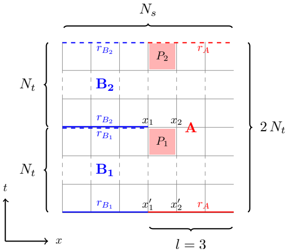
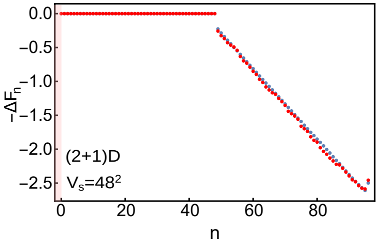
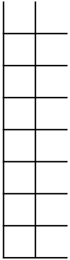
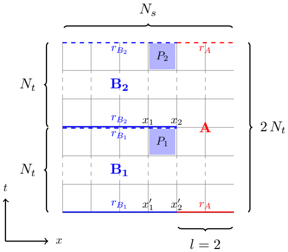






Entangling surface deformation method

How can we avoid (huge) free energy barriers?



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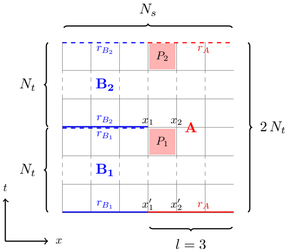

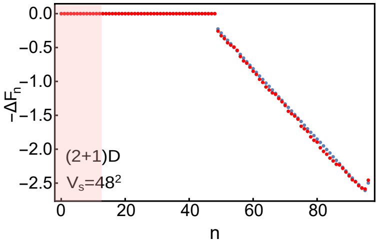

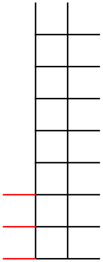

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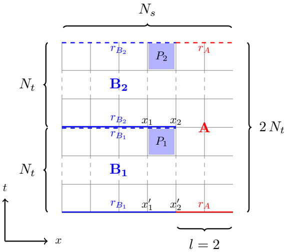

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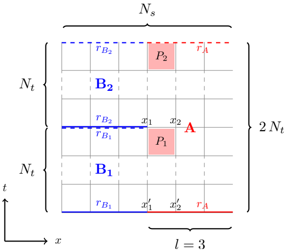

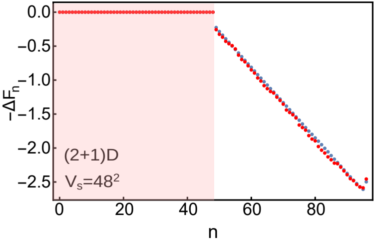

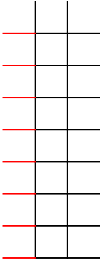

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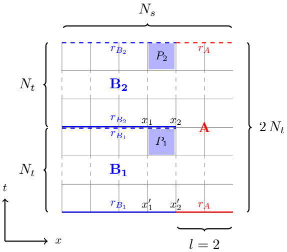

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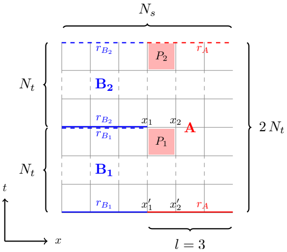

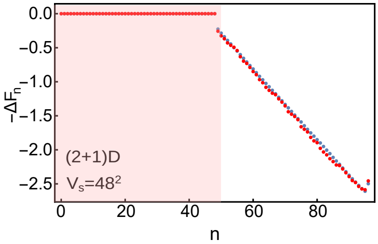

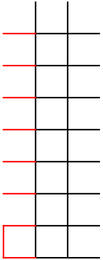

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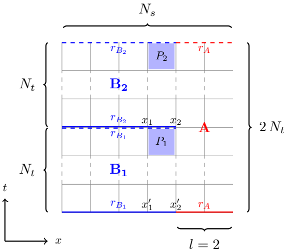

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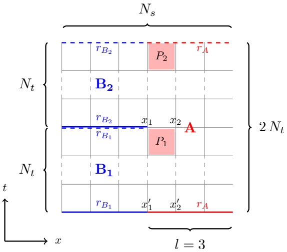

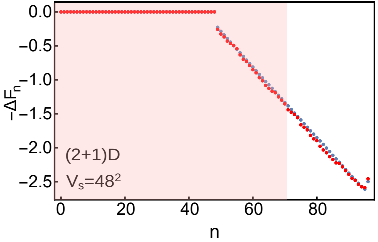

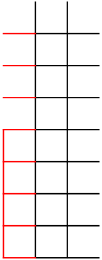

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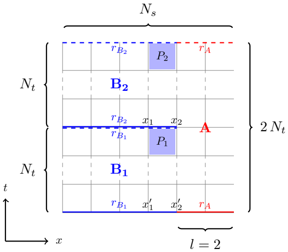

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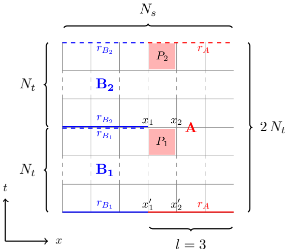

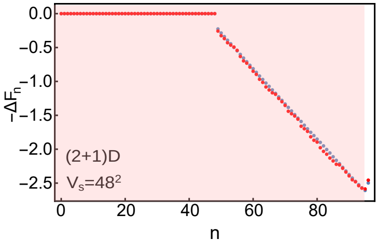

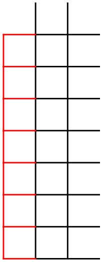

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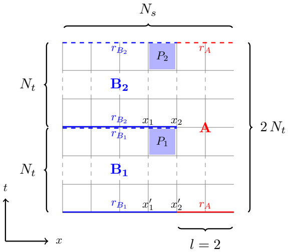

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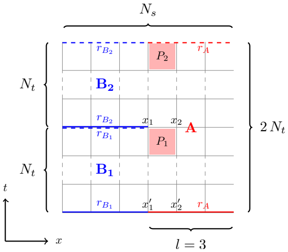

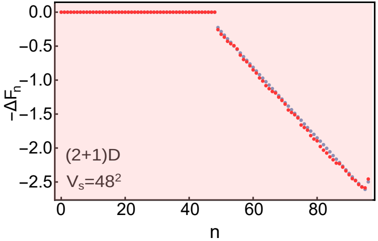

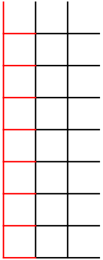

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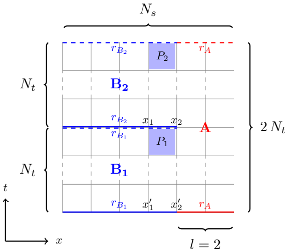

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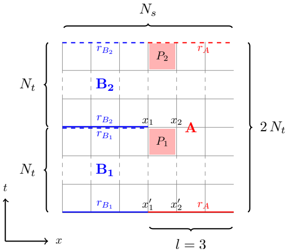

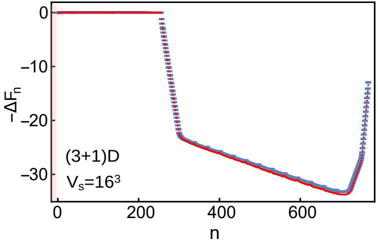

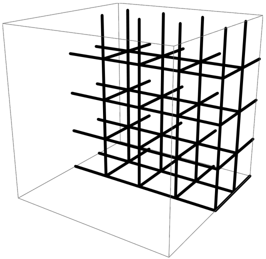

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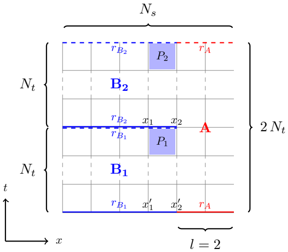

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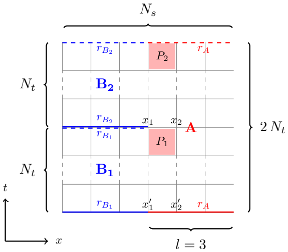

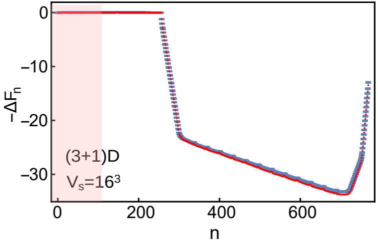

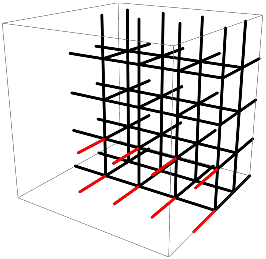

Entangling surface deformation method

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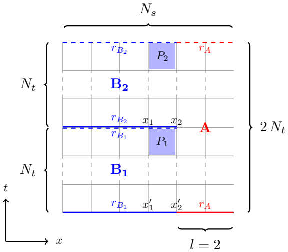

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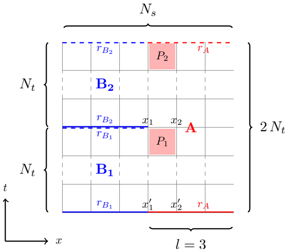

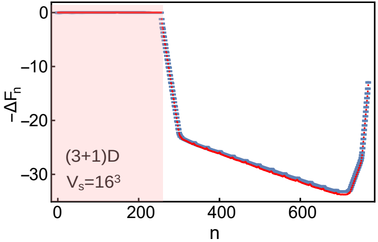

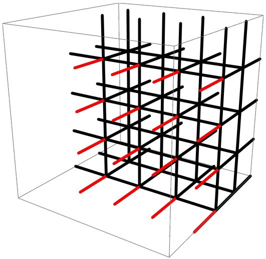

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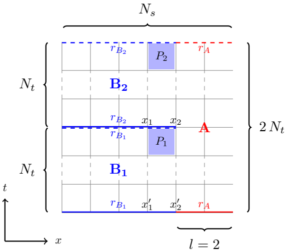

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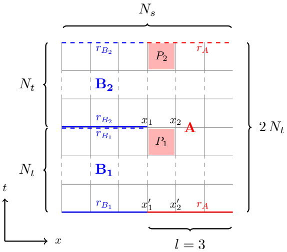

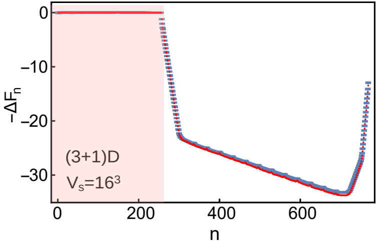

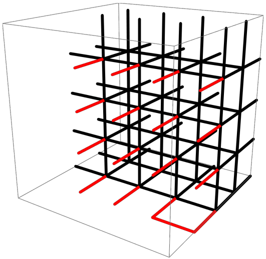

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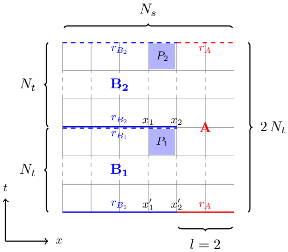

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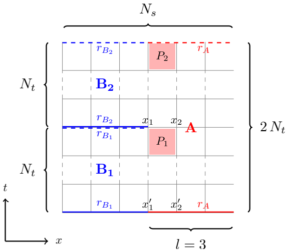

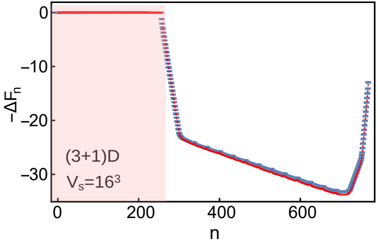

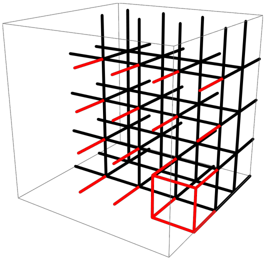

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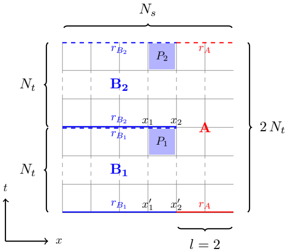

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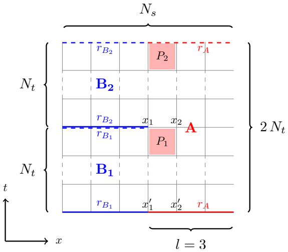

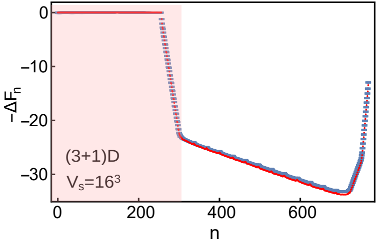

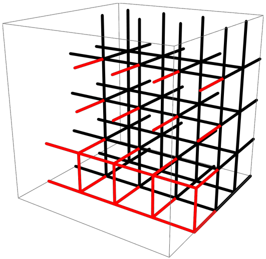

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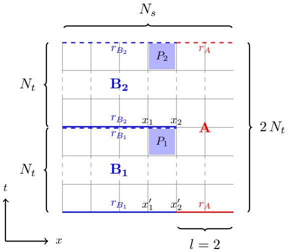

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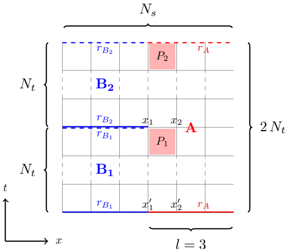

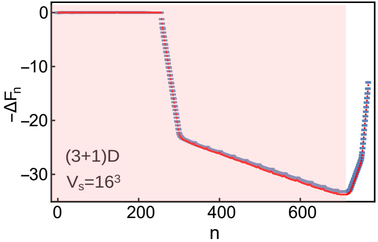

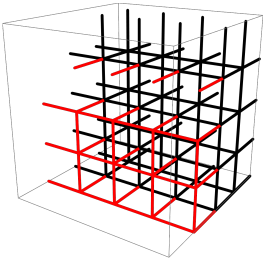

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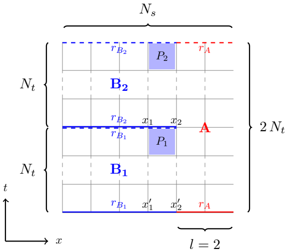

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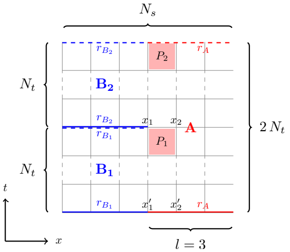

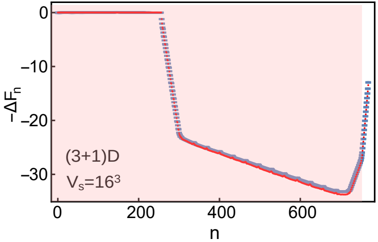

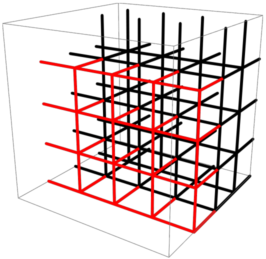

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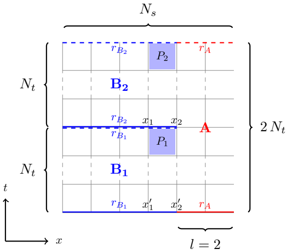

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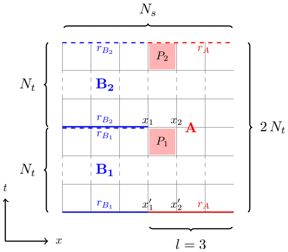
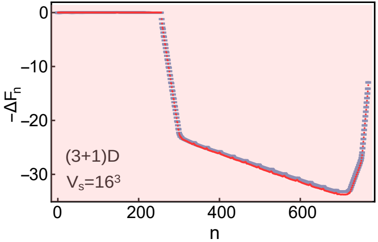
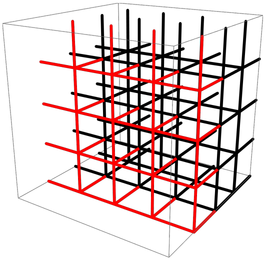


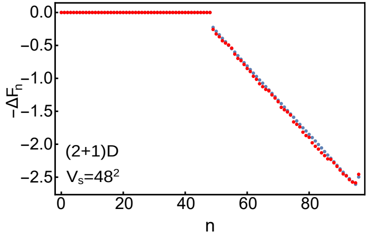


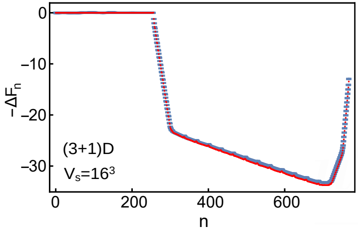


Entangling surface deformation method

Free-energy plateau

- Why does the free energy initially not change?





Free-energy plateau

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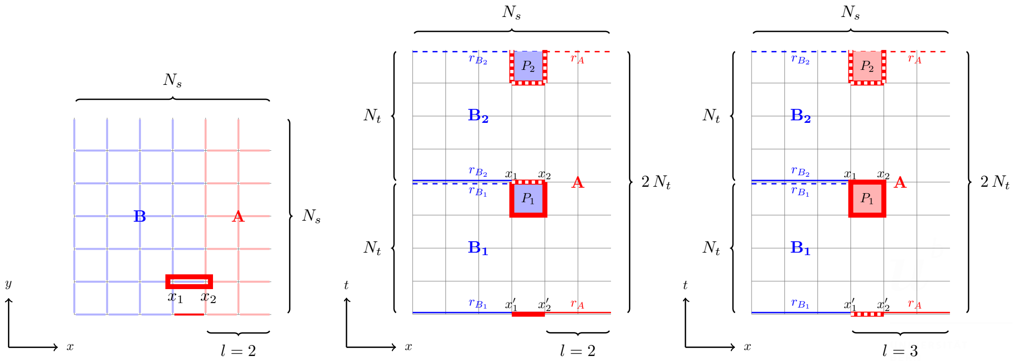

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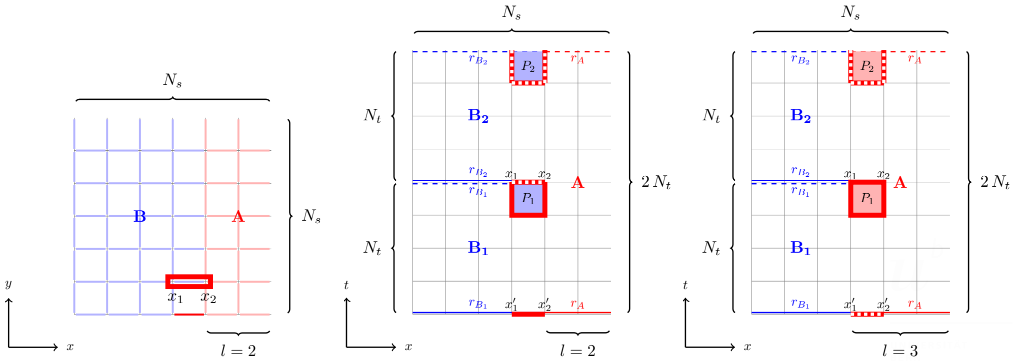

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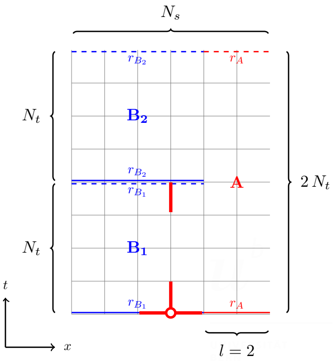

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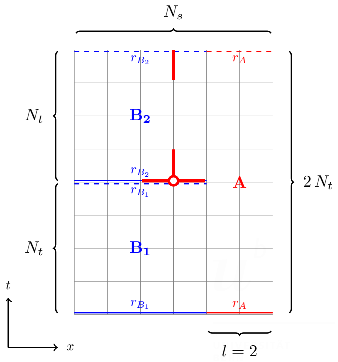

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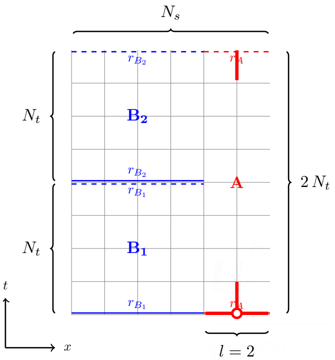

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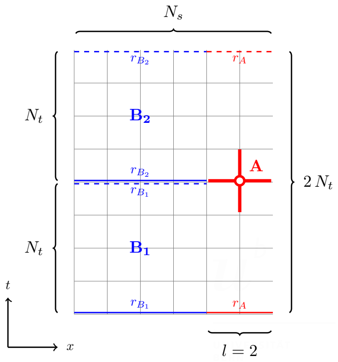

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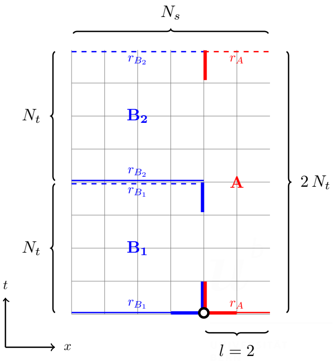

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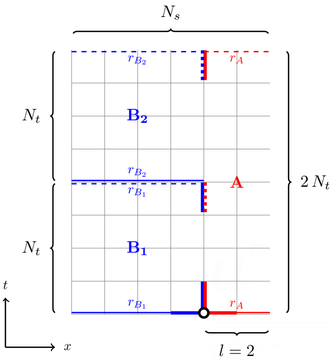

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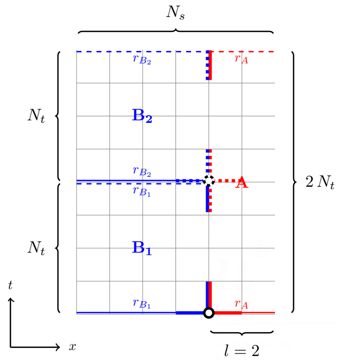

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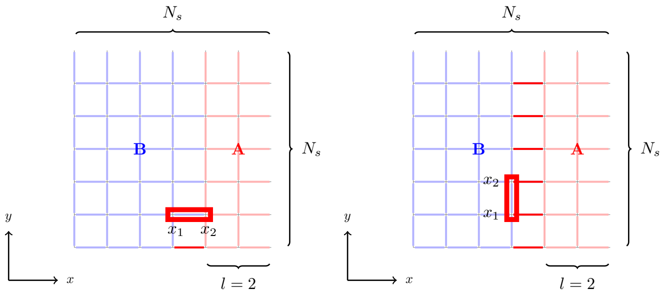

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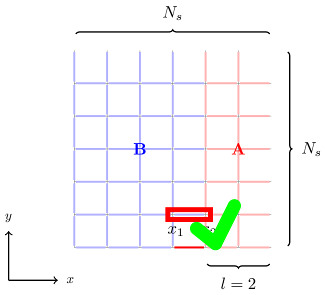

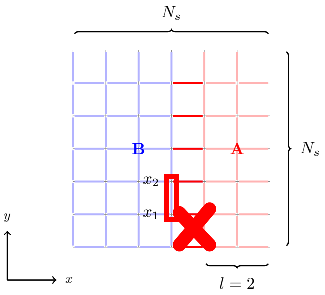

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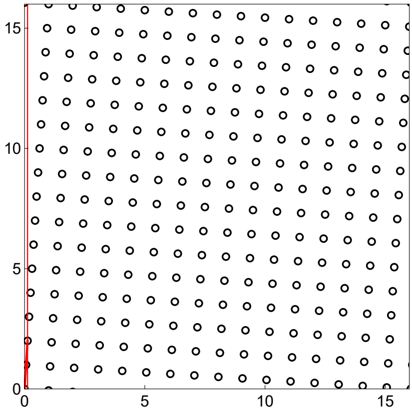

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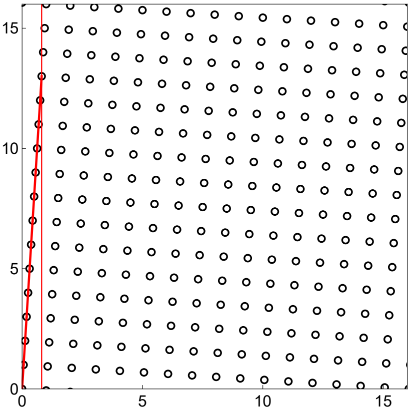

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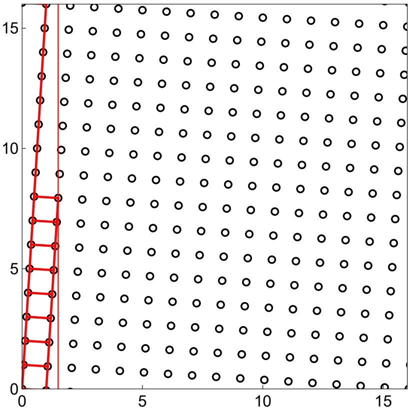

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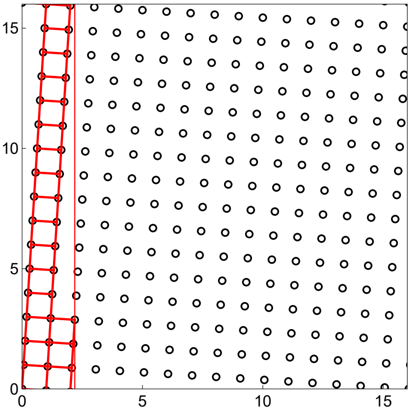

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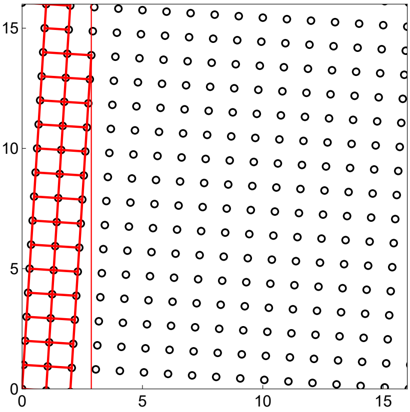

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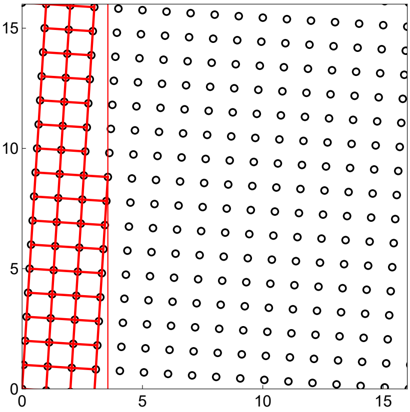

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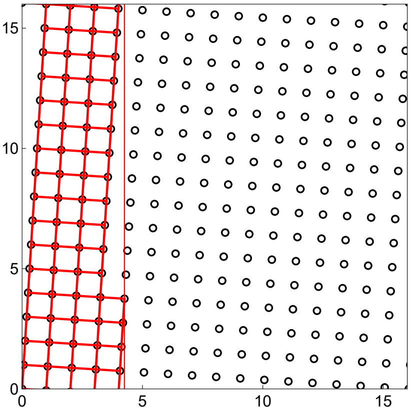

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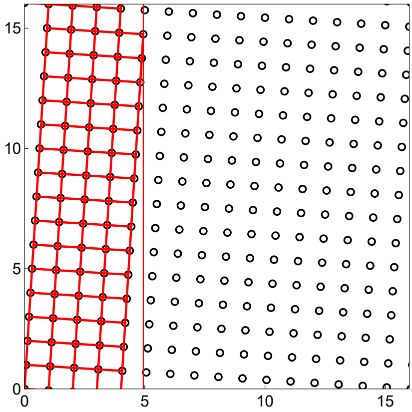

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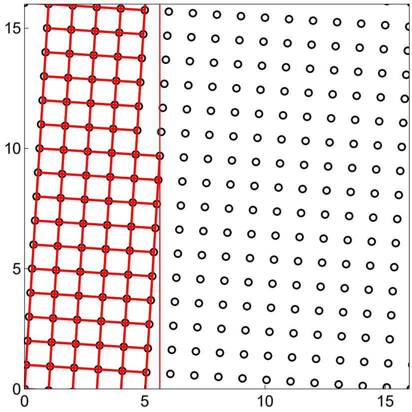

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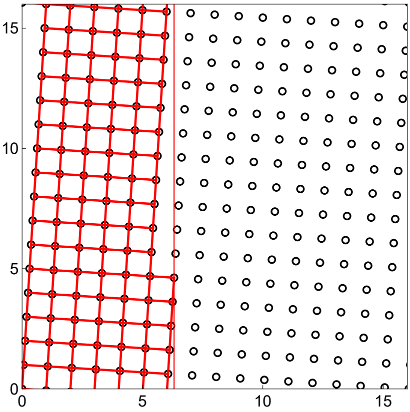

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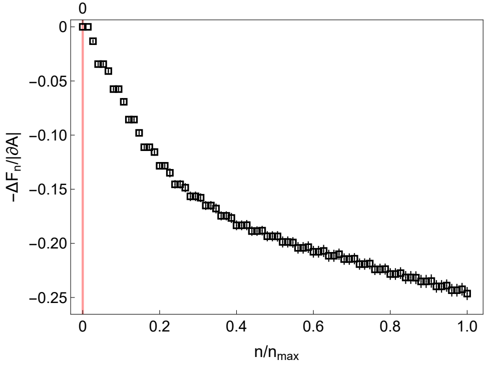

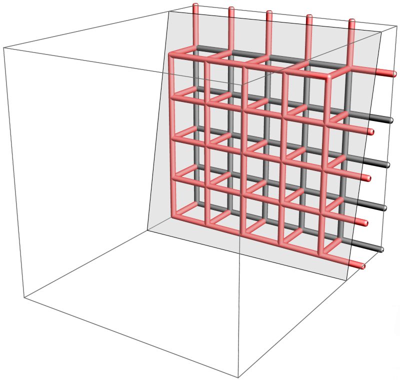

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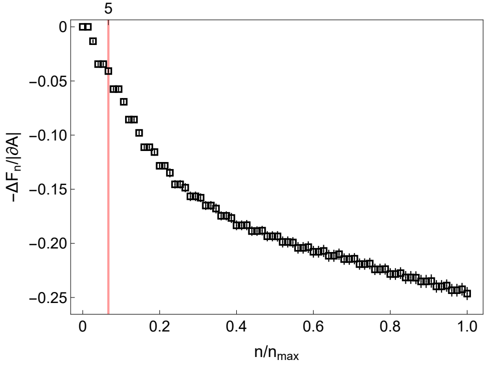

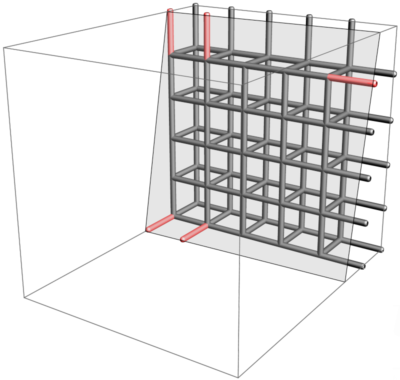

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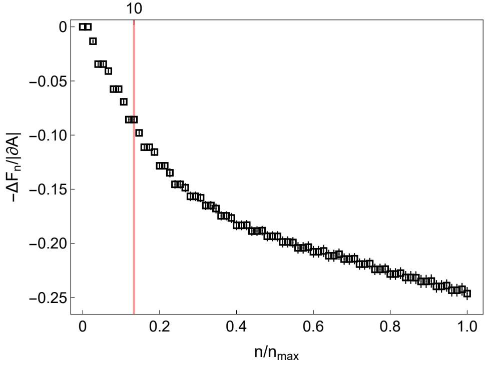

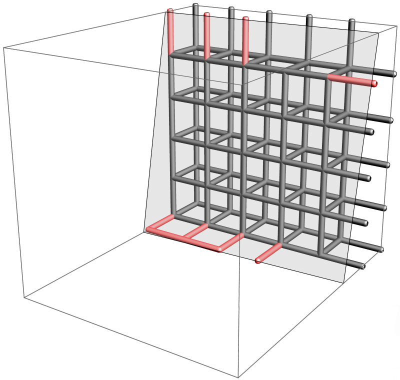

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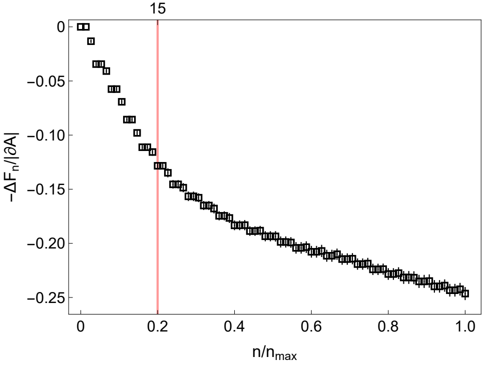

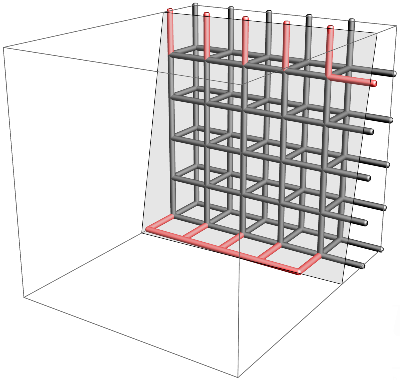

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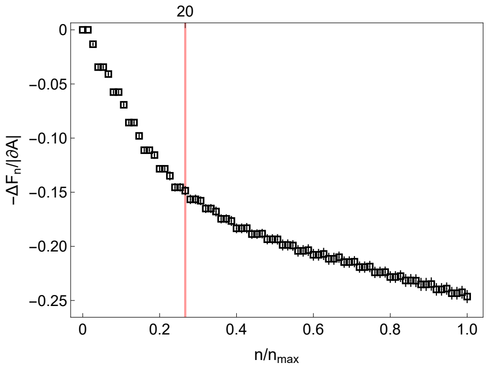

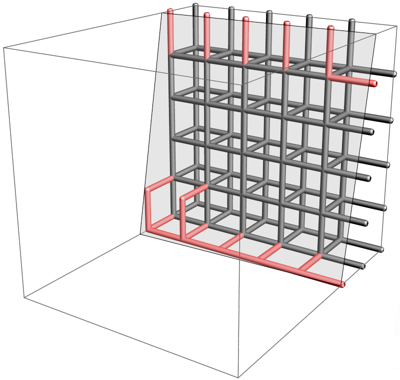

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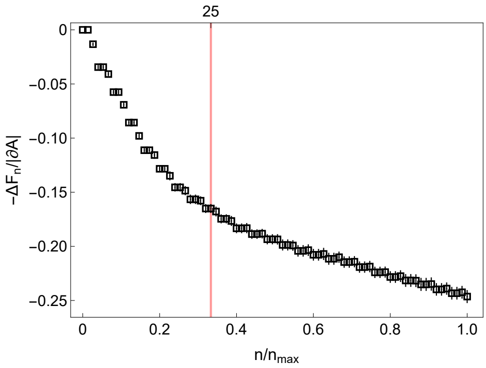

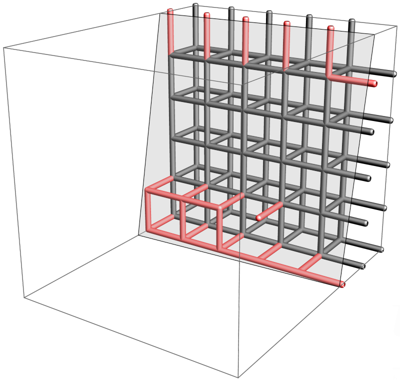

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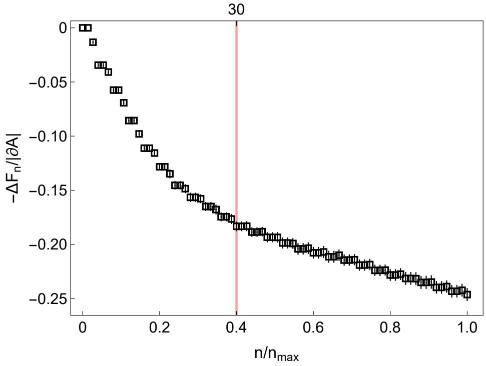

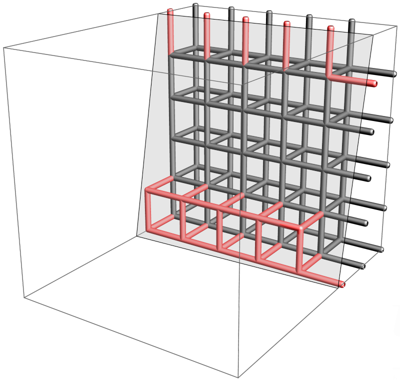

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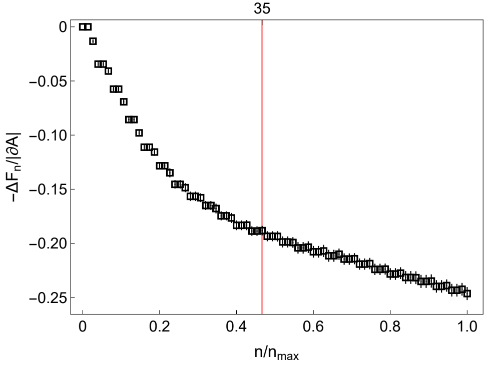

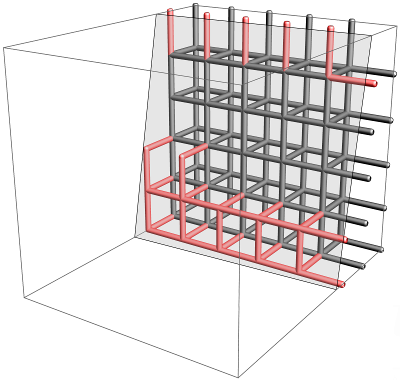

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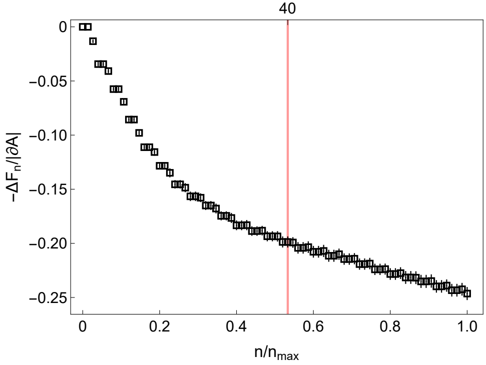

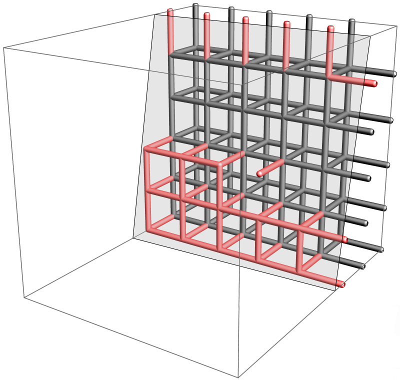

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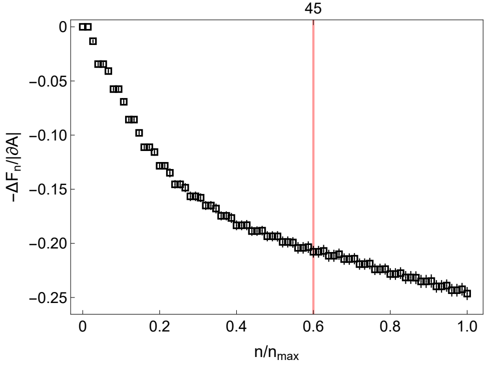

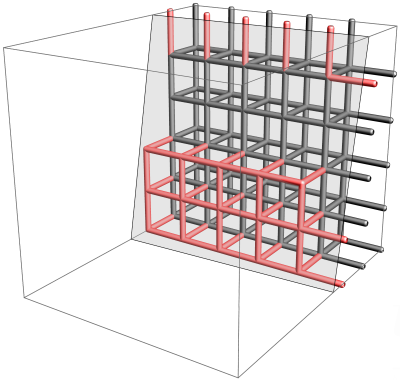

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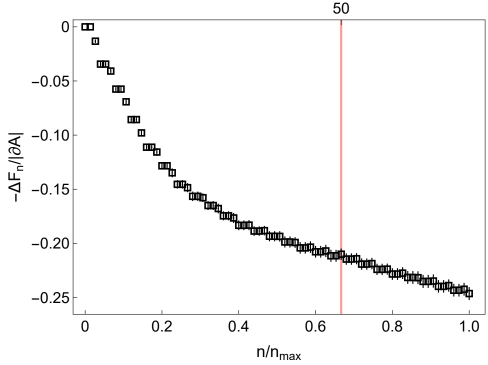

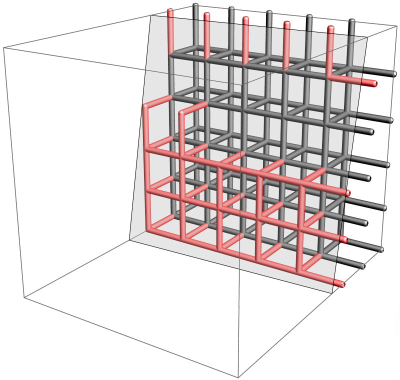

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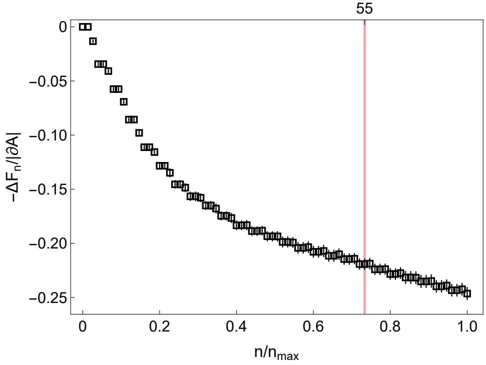

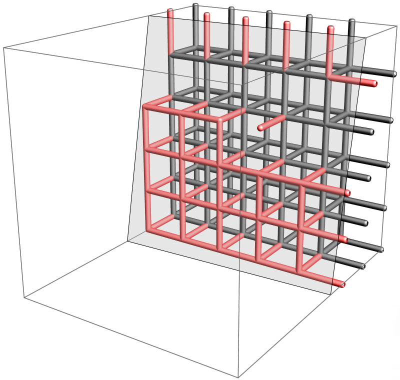

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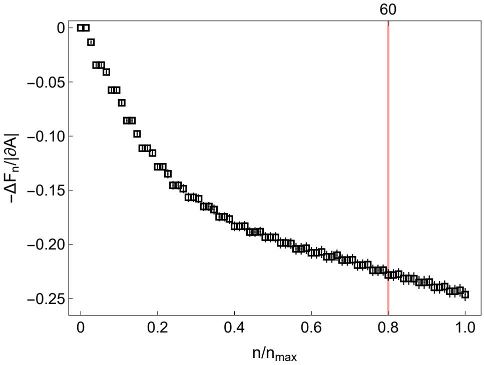

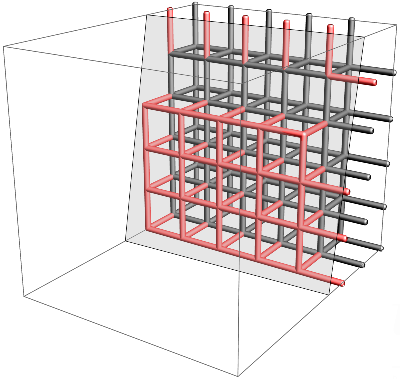

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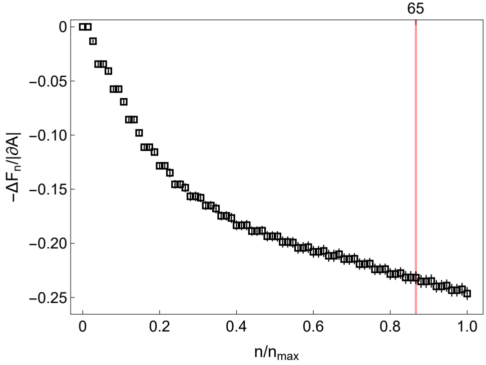

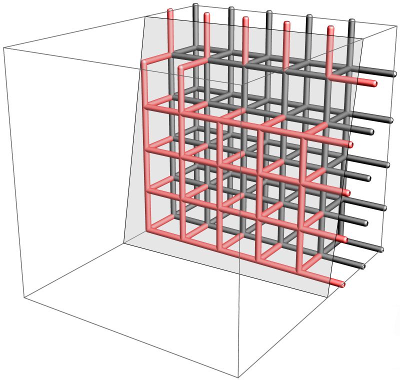

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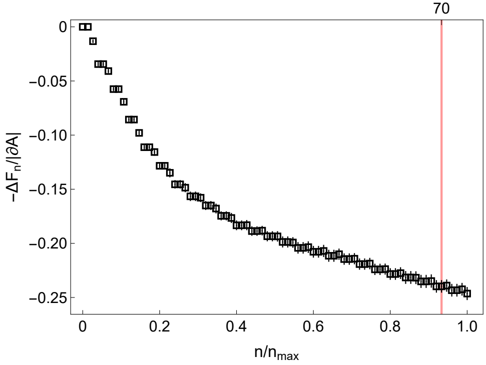

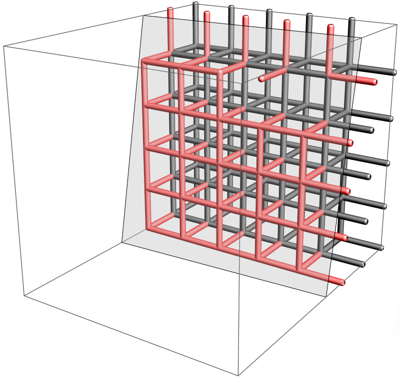

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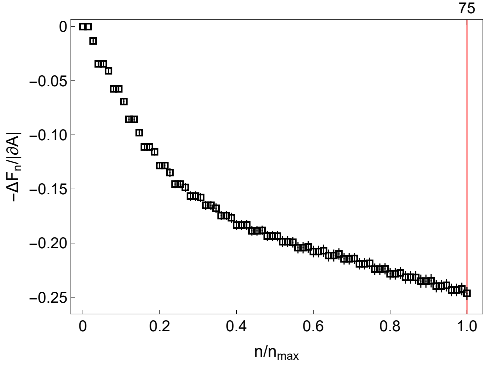
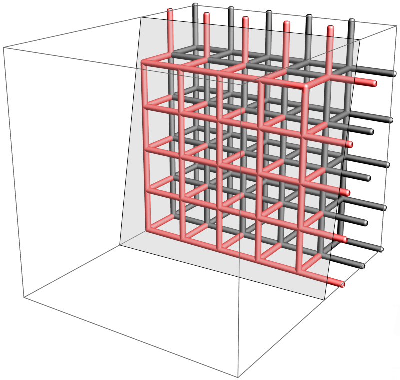




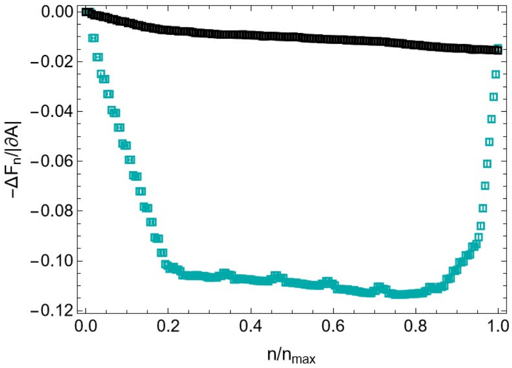
Entangling surface deformation method

Can we avoid free energy barriers completely?

- Yes ➔ use "tilted lattice"

  ➔ SU(2) in (3+1) dimensions:

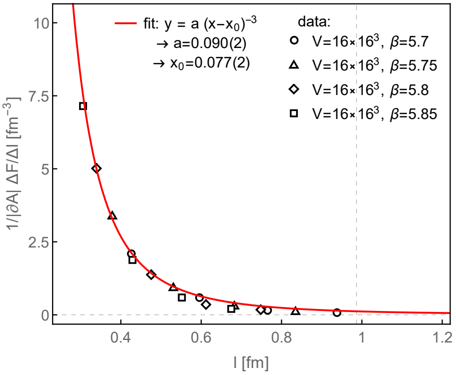
  comparison of boundary update methods: non-tilted lattice ⟷ tilted lattice

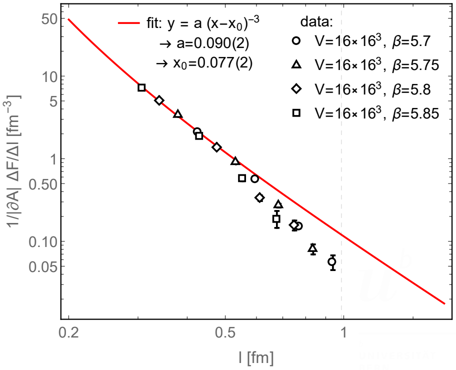


Results

Results in 4D **[arXiv.2211.00425]**

- Entanglement entropy change as function of entangling region width l for SU(3) on $N_s^3 \times 2 \cdot N_t$ lattice with $N_s = N_t = 16$, $\beta \in \{5.7, 5.75, 5.8, 5.85\}$.





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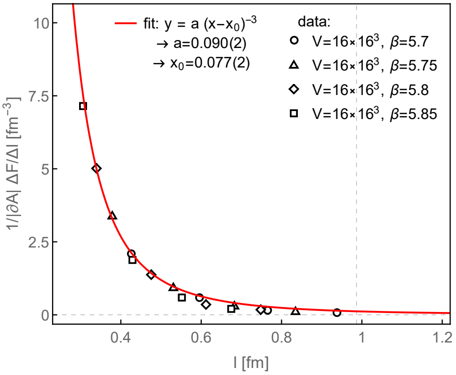

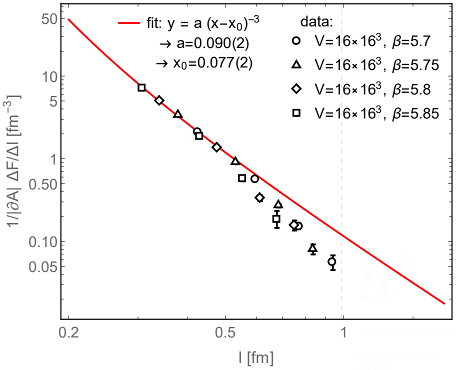

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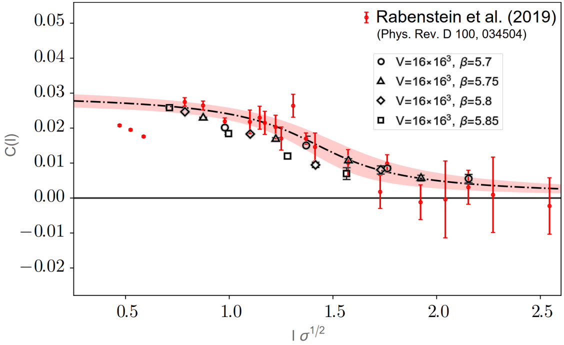

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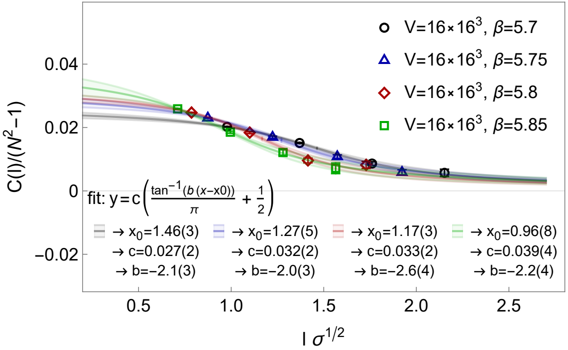

Relation to thermal entropy **[arXiv:2304.08949]**

- Thermodynamic entropy on the lattice

  Lattice free energy: $F_L(N_t, V, N) = -\log(Z(N_t, V, N)) = N_t\, F(T(N_t), V, N)$

  (spatial lattice volume V, temporal lattice size N_t, some charge N)

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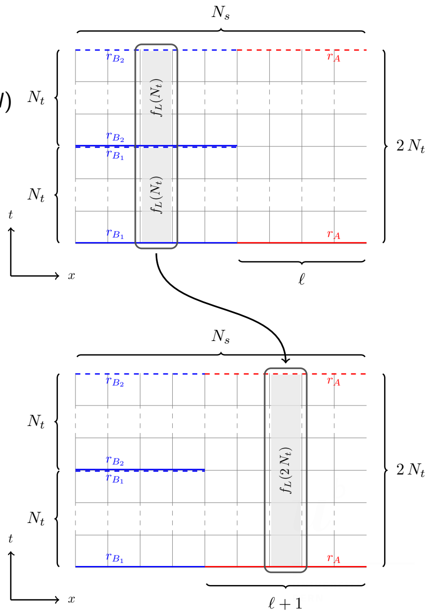

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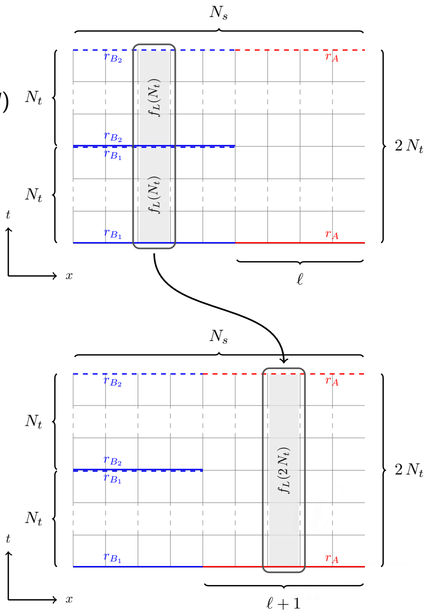

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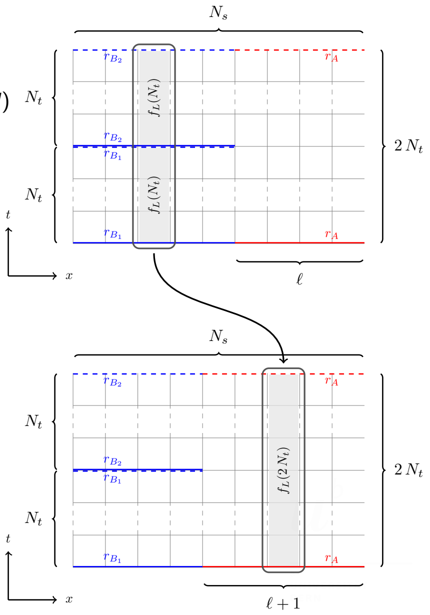

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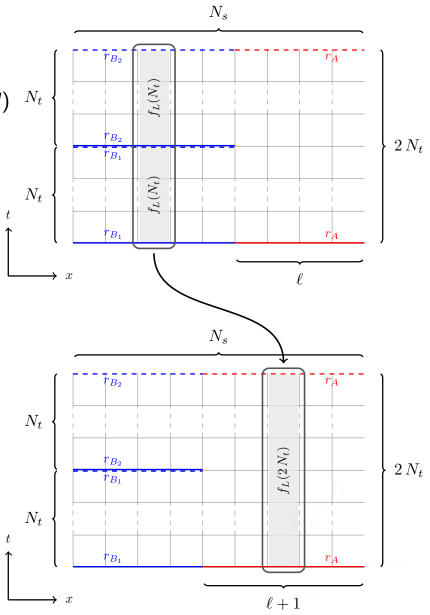
  ➔ Replica trick can be used to determine thermal entropy

  (without having to know F_L or other intergration constants)



Relation to thermal entropy [arXiv:2304.08949]

- Test in (2+1) dimensions at high temperature:
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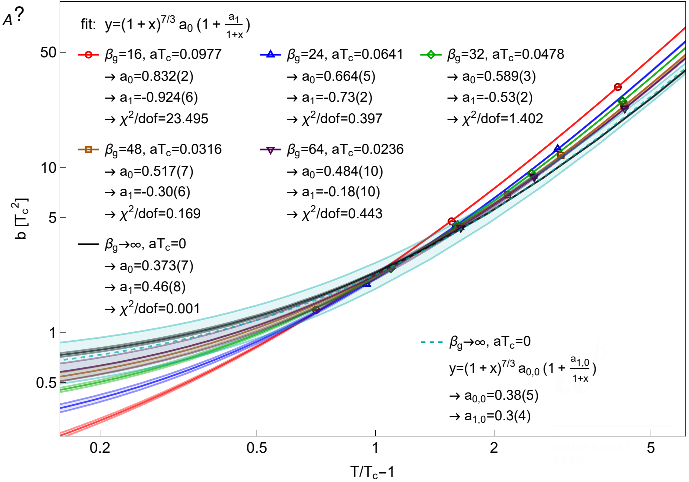

Conclusions & outlook

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- New method to determine entanglement measures (Rényi and entropies) in SU(N) lattice gauge theories.
  - No more free energy barriers when using "tilted lattice".
    - ➔ significant error reduction possible.
  - Comparison with literature results promising.
  - Replica trick can be used to compute thermal entropy.

Outlook

- Application to further cases:
  - SU(N), $N = 2, 3, 4, 5, \ldots, ?$, $d = 3, 4$, $T = 0$, $T \neq 0$
  - different entangling region shapes; alternative entropy measures?
  - "metric reconstruction" (holography) for SU(2), SU(3)?
- improved simulation algorithm for "tilted lattice".

Thank you!