Algebraic Geometry in the Daylight

Aug 28, 2023, 9:00 AM → Aug 30, 2023, 5:00 PM Europe/Oslo
KE D-302

David Ploog (University of Stavanger) , Helge Ruddat (University of Stavanger) , Martin Gulbrandsen (University of Stavanger)

Description We are running a three day conference at the University of Stavanger about new developments in algebraic geometry. The speakers are

Javier Bobadilla (Bilbao)
Alessio Corti (London)
Lutz Hille (Münster)
Michal Kapustka (Kraków)
Martí Lahoz (Paris)
Georg Oberdieck (Stockholm)
John Christian Ottem (Oslo)
David Rydh (Stockholm)
Kris Shaw (Oslo)
Vivek Shende (Odense)
Sofia Tirabassi (Stockholm)

If interested in participating, please register as soon as you can.

We are planning an excursion and hike onto Preikestolen on Tuesday afternoon.

Registration Registration for Algebraic Geometry in the Daylight 28 Aug - 30 Aug

Participants ALESSANDRO PASSANTINO, Alessio Corti, Alex Nielsen, Alexei Latyntsev, Alexious Malata, Anba-Maria Raukh, Andres David Gomez Villegas, David Ploog, David Rydh, Eirik Eik Svanes, FENGLONG YOU, Georg Oberdieck, Helge Ruddat, Javier Bobadilla, Javier Murgas, John Christian Ottem, Kelvin Malunga, Kris Shaw, Lutz Hille, Maria Azam, Marti Lahoz, Martin Gulbrandsen, Michal Kapustka, PRAVEEN PANDEY, Sofia Tirabassi, Tyson Ritter,  Vegard Undheim, Vivek Shende, Xianyu Hu

Contact helge.ruddat@uis.no

MONDAY, AUGUST 28

9:00 AM → 9:55 AM **Symplectic structures at radius 0 associated with degenerations and their applications** 55m KE D-302

Speaker: Javier Bobadilla

10:30 AM → 11:25 AM **Tropical degenerations and stable rationality** 55m KE D-302

Speaker: John Ottem (Oslo)

1:00 PM → 1:55 PM **Cohomologically tropical varieties** 55m KE D-302

The cohomology of the complement of a hyperplane arrangement can be recovered from its tropicalisation. This talk asks what which other subvarieties of the torus does this hold. We show that the tropicalisation knows the cohomology of the variety in a strong sense if and only if it satisfies local tropical Poincaré duality and the original variety is what we call wunderschön. Following the work of Itenberg, Katzarkov, Mikhalkin, and Zharkov, we have that tropicalisations of families of varieties which locally satisfy these two conditions recover information about the mixed Hodge structures of the family.
This is joint work with Edvard Aksnes, Omid Amini, and Matthieu Piquerez

Speaker: Kris Shaw

2:15 PM → 3:10 PM **Non-reductive geometry in the daylight** 55m KE D-302

Speaker: David Rydh

4:00 PM → 4:55 PM **Gromov-Witten theory of the Enriques surface** 55m KE D-302

Speaker: Georg Oberdieck

TUESDAY, AUGUST 29

9:00 AM → 9:55 AM **Towards homological mirror symmetry** 55m AR Ø-110

Survey of my ongoing work with Benjamin Gammage.

Speaker: Vivek Shende

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10:30 AM → 11:25 AM **Twisted FM partners of ordinary K3 surfaces** 🕓 55m
Speaker: Sofia Tirabassi AR Ø-110

WEDNESDAY, AUGUST 30

9:00 AM → 9:55 AM **How to make log structures** 🕓 55m
Speaker: Alessio Corti KE E-101

10:30 AM → 11:25 AM **Polynomial invariants for full exceptional sequences** 🕓 55m
Speaker: Lutz Hille KE E-101

1:00 PM → 1:55 PM **Some derived equivalent hyperkahler fourfolds of Picard number 1** 🕓 55m
Speaker: Michal Kapustka AR G-001

2:15 PM → 3:10 PM **Cohomological rank functions on abelian surfaces via Bridgeland stability** 🕓 55m
Speaker: Martí Lahoz AR G-001

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Javier Bobadilla: Lag fibratn on max CY deg

Let $X^\circ \to \Delta^\circ$ be a maximal CY degeneration

Consider $[\omega]$ class in the ample cone

Yau's solution of CY conjecture: $\exists\, z \in \Delta^\circ$

$\exists!\ \omega_z^{CY}$ Ricciflat metric st. $[\omega_z^{CY}] = [\omega]$

$g_z^{CY} := \omega_z^{CY}(-, J-)$

$\tilde{g}_z^{CY}$ Rescaling to diameter 1

Conjecture $\displaystyle \lim_{\overrightarrow{z \to 0}} (X_z, \tilde{g}_z^{CY}) = (\bar{Y}, g_{\bar{Y}})$

$(Y, g_Y) \subset (\bar{Y}, g_{\bar{Y}})$

$\bar{Y} \setminus Y$
Hausdorff
codim ≥ 2

Riemannian manifold with integral affine structure $g_Y = \dfrac{\partial^2 K}{\partial y_i \partial y_j} = (g_{ij})$

$y_1 \dots y_d$ affine coordinates

$\det g_{ij} = \text{const}$ Real MA equ.

$X_z = X_z^{sm} \amalg X_z^{sing}$

$X_z^{sm} \to Y$ fibration by Lagrangian tori

$(X_z, X_z^{sm}) \to (\bar{Y}, Y)$

Further part: the fibration should "resemble asymptotically" the canonical one over a affine manifold.

Consider $X \to \Delta$ snc model such that

$X|_{\Delta^\circ} = X^\circ$ $\qquad D = \sum_{i=1}^{N} m_i D_i \rightsquigarrow$ dual complex

$\Delta = \{ \sum_i w_i = 1 \} \subseteq \mathbb{R}^N$
$\cup$
Δ_D $\qquad$ Consider Ω a meromorphic n-form

$\operatorname{div}(\Omega) = \sum_i a_i D_i$ $\qquad$ assume $\min\{a_i\} = 0$

$Sk \subset \Delta_D$ subcomplex generated by essential vertex

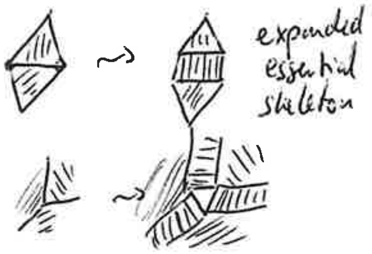
↖ zero discrepancy

Results of Mustata, Nicaise, Xu, Yu (Kollar?)

- Sk has pw affine structure as being part of a simplex
  and Riemannian metric $g = \sum_i (dw_i)^2$
  satisfies real MA equ (piecewise)
- Sk is independent of the model, together w pw affine structure
- $\exists$ snc model st. if $a_i = 0 \Rightarrow m_i = 1$
- if $I \subset \{1, \dots N\}$ s.t. $\# I = n$ submax face

(X_I, X_I°) $\qquad X_I = \bigcap_{i \in J} D_i$
$\shortparallel$
$(\mathbb{P}^1, \mathbb{C}^*)$ $\qquad X_I^\circ = X_I \setminus \bigcup_{j \notin I} X_j$

Essential skeleton



Theorem (Bobadilla, Pelka)

Let $X^\circ \to \Delta^\circ$ maximal CY degeneration

Pick a model with the stated properties

ω_X Kähler form

There exists a family ω_z of Kähler forms $z \in \Delta^\circ$ with the following properties

a) $[\omega_z] = [\omega_X|_{X_z}]$ $\qquad \omega_z = \omega_X|_{X_z} + dd^c \phi_z$

b) $g_z := \omega_z(-, J-)$

$\tilde{g}_z$ rescaling to diam 1 $\qquad \lim (X_z, \tilde{g}_z) = (Sk, \sum dw_i^2)$

c) $\forall z \in D_z^*\ \exists\, X_z^{sm} \subset X_z$
and a fibration $X_z^{sm} \longrightarrow \overline{Sk}$ codim $= 2$

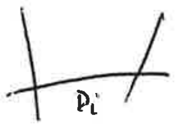
d) Ω hol vol form
scaled st $\displaystyle\int_{X_z} \Omega \wedge \bar{\Omega} = 1$ $\qquad \displaystyle\lim_{z \to 0} \int_{X_z^{sm}} \Omega \wedge \bar{\Omega} = 1$

asymptotically
special Lagrangian $\qquad \displaystyle \neq \int_{X_z^{sm}} \omega_z^n$

Main steps of proof

"Symplectic A'Campo space"

$X \xrightarrow{f} \Delta \quad \omega_X \qquad \hat{r}_i : X \to \mathbb{R}$

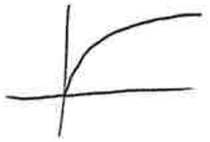


adapted chart

$f(z_1 \dots z_n) = \prod z_i^{m_i}$

$\hat{r}_i = |z_i| \qquad \hat{s}_i = \log \hat{r}_i$

$\eta(s) = \frac{1}{1-\log(s)}$



similar to radial coordinate $\to$ $\hat{t}_i = -\frac{1}{m_i \hat{s}_i}$

$t = -\frac{1}{\log|t|}$

tropical coordinates $\to$ $\hat{\omega}_i = \frac{t}{\hat{t}_i}$

$s_i = \log|z_i|$

$t_i = -\frac{1}{m_i s_i}$

$w_i = t/t_i$

$\sum w_i = 1$

$\hat{u}_i := \mu(\hat{\omega}_i)$

$v_i := t_i - u_i$

↖ hybrid coordinate

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr] & \Gamma_\mu \arrow[d] & \\
(X,D)_{\log} \arrow[r, "\pi_A"] \arrow[d] & (X,D) \arrow[r, dashed, "\mu", "\hat{u}_1 \dots \hat{u}_n"'] \arrow[d] & (\mathbb{R}^N_{\geq 0}) \\
(\Delta,0)_{\log} \arrow[r] \arrow[d, equal] & (\Delta,\{0\}) & \\
\mathbb{C}_{\log} & &
\end{tikzcd}$$

Prop: $\exists\, C^\infty$ atlas in A s.t.

1) $A \longrightarrow \mathbb{C}_{\log}$ is a smooth submersion
2) $A \xrightarrow{\pi_A} X$ is C^∞
3) $\hat{w}_i$ are C^∞
4) $\hat{v}_i$ are C^∞
5) $d^c \hat{s}_i$ =: extends to a smooth 1-form in A

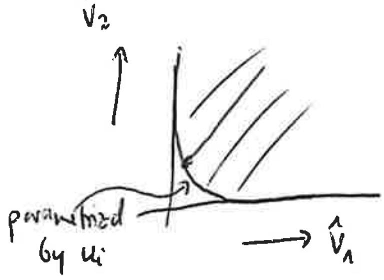
$s_i = \log|z_i|$

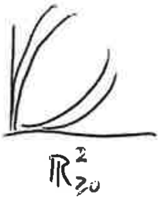
$d^c s_i = d\theta_i$

Example $\quad f(z_1, z_2) = z_1 z_2$

Kato-Nakayama Space $\qquad (S^1)^2 \times (\mathbb{R}_{\geq 0})^2 \leftarrow A$

angular part ↗ $\qquad$ ↖ radial part





$$\underbrace{\varepsilon\left(\underbrace{\sum_{i=1}^{N} d(\hat{v}_i \hat{\alpha}_i)} + \overset{(1-q(t))}{\vee}\sum_{i \in I_{ess}} -\frac{1}{2t} dd^c(w_i)^2\right) + \pi^*\omega_X}_{=:\, \omega_t}$$

$q(t)$ vanishing fast

allows many Lagrangians via connection

$v_i = t_i - \eta(w_i)$

$d\theta_i$

$dv_i \wedge d\theta_i$

$d(\hat{v}_i \hat{\alpha}_i) = dv_i \wedge d\theta$ + trash terms

Toric degenerations and stable rationality

John Christian Ottem

$\boxed{D}$ X is rational if is birational to projective space

$\boxed{D}$ X is stably rational if $X \times \mathbb{P}^n$ is rational for some $n>0$

$\boxed{D}$ X, Y are stably birat. if $X \times \mathbb{P}^n \sim_{bir} Y \times \mathbb{P}^m$

(Main application) $\boxed{\text{Th A}}$ The very general quartic 5-fold $X = Z(F) \subset \mathbb{P}^6$ is not stably rational

$\boxed{\text{Th B}}$ A very general $(2,3)$ c.i. $X \subset \mathbb{P}^6$ $X = Z(Q, C)$ is not stably rational

$\boxed{ex}$ Cubic 3-folds

$\boxed{ex}$ Schreieder $X \subset \mathbb{P}^n$ degree d $(n \geq 5)$ $d \geq \log_2 n+2 \Rightarrow X$ stably irrational

- Specialization of stable rationality (Nicaise-Shinder–Kontsevich-Tsch..)
- Tropical
- irrationality of varieties

SB_F = set of stable birat. types

$X/_F \rightsquigarrow [X] \in SB_F$ $\quad$ $\mathbb{Z}[SB_F]$ is a ring $\quad [X][Y] = [X \times_F Y]$

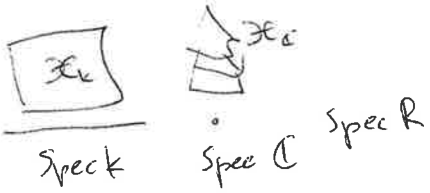
Larsen-Lunts $\quad \dfrac{K_0(Var/_F)}{(\mathbb{L})} \xrightarrow{\sim} \mathbb{Z}[SB_F] \quad \mathbb{L} = [\mathbb{A}^1]$

Notation $\quad K = \bigcup_{m>0} \mathbb{C}((t^{1/m})) \qquad R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]]$

Th Nic.-Shinder : $\exists$ ring map $\mathrm{Vol} : \mathbb{Z}[SB_K] \to \mathbb{Z}[SB_{\mathbb{C}}]$

s.t. $\mathcal{X}/_R$ proper + semistable

$$\mathrm{Vol}([\mathcal{X}_K]) = \sum_{\substack{J \subset I \\ J \neq \emptyset}} (-1)^{|J|-1} [X_J]$$



$X_J = X_{j_1} \cap \ldots \cap X_{j_r}$

Consequences:

- $\mathrm{Vol}([\mathrm{Spec}\,K]) = \mathrm{Spec}\,\mathbb{C}$ ⇝ obstruction to stable rationality:

  if $\sum (-1)^{|J|}[X_J] \neq [\mathrm{Spec}\,\mathbb{C}]$ — nonnegligible?

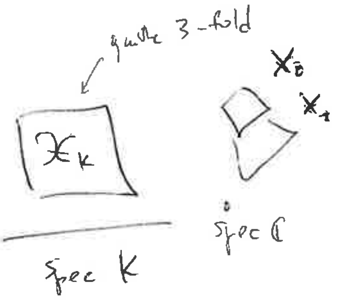
- $\mathcal{X}/R$ proper + smooth $\Rightarrow \mathrm{Vol}([\mathcal{X}_K]) = [\mathcal{X}_0]$

$\boxed{\text{ex}}$ (Voisin) A very general double quartic solid is stably irrational

$X \to \mathbb{P}^3$
$y^2 = f_4(x_0 .. x_3)$ ⇝ … to the Artin-Mumford example

X_0 $\mathrm{Tors}\, H^3(\tilde{X}_0, \mathbb{Z}) \neq 0$

Quartic 5-folds



$$\mathcal{X} = \mathrm{Proj}\ \frac{R[x_0 .. x_6, y]}{(yt - x_5x_6;\ y^2 - F(x_0 ... x_6))}$$

v.g with conditions

X_0: $y^2 = F(x_0 .. x_6)$, $x_5 = 0$

X_1: $y^2 = F(..)$, $x_6 = 0$

$\boxed{\text{we will choose } X_0 \simeq X_1}$

X_{01} $y^2 = F(x_0 .. x_6)$ $x_5 = x_6 = 0$ v.g. double quartic 4-fold ! → they are irrational by HPT

$$\mathrm{Vol}([\mathcal{X}_K]) = [X_0] + [X_1] - [X_{01}] = 2[X_0] - [X_{01}]^{\neq [\mathrm{Spec}\,\mathbb{C}]} \neq [\mathrm{Spec}\,\mathbb{C}]$$

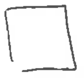
(because $\mathbb{Z}[SB_{\mathbb{C}}]$ is free)

Toric degenerations

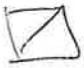
Δ lattice polytope ⇝ $Y = Y(\Delta)$ polarized toric variety

P regular subdivision ⇝ degeneration of Y into $\bigcup_{p \in P} Y_p$

$\boxed{\text{ex}}$



$\longrightarrow$



$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ $(\mathbb{P}^2, \mathcal{O}(1)) \cup (\mathbb{P}^2, \mathcal{O}(1))$

$\boxed{\text{ex}}$

$\longrightarrow$ $Y_0 \cup Y_1$

$Y_0 \cap Y_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$

$[X_1] + \dots + [X_n] - [\ ] + \dots \neq [\mathrm{Spec}\,\mathbb{C}]$

$\boxed{\text{ex}}$ Quartic 5-fold $(\mathbb{P}^6, \mathcal{O}(4))$ ⇝ 4 toric 6-folds

polytope of (2,2) $\mathbb{P}^2 \times \mathbb{P}^3$ ⇝ irrational by HPT

lattice width 1 ⇒ $A + xB = 0$ ⇝ rational

Nonreductive geometry in daylight

David
Rydh
char 0

G reductive $\curvearrowright \operatorname{Spec} A$

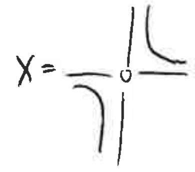
$$X/\!\!/G = \operatorname{Spec} \underbrace{A^G}_{\text{finite gen.}}$$

& unique closed point

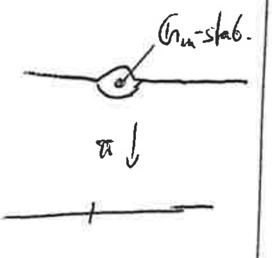
$$\begin{tikzcd} X \arrow[d] \arrow[dd, bend right, "\text{good quotient}"'] \\ {[X/G]} \arrow[d, "\text{good moduli space (gms)}"] \\ X/\!\!/G \end{tikzcd}$$

good quotient

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $+1, -1$

$X =$ 

$$\begin{tikzcd} {[X/\mathbb{G}_m]} \arrow[d] \\ X/\!\!/\mathbb{G}_m \end{tikzcd}$$



$t = xy$

$\pi^{-1}(0)$
unique closed point

projective GIT

$G \underset{\text{red}}{\subset} X$ $\mathcal{L} \in \operatorname{Pic}^G(X)$ ample

$\rightsquigarrow X^{ss} \subset X$ open

$$X/\!\!/_{\mathcal{L}} G = X^{ss}/\!\!/G = \operatorname{Proj}\underbrace{\Big(\bigoplus_{d \geq 0} \Gamma(X, \mathcal{L}^d)^G\Big)}_{\text{f.g.}}$$

locally just affine GIT

$$\begin{tikzcd} X^{ss} \arrow[r, phantom, "\subset"] \arrow[d] & X \arrow[ddl, dashed, bend left] \\ {[X^{ss}/G]} \arrow[d, "\text{gms}"] & \\ X^{ss}/\!\!/G & \end{tikzcd}$$

Def (Alper '08) $\pi : \mathcal{X} \to X$ is gms if

(1) $\pi_* \, \operatorname{QCoh}(\mathcal{X}) \to \operatorname{QCoh} X$ is exact

(2) $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$

Rmk: In affine GIT $\operatorname{Mod}_A^G \to \operatorname{Mod}_{A^G}$
$M \longrightarrow M^G$ exact

§ Properties of gms $\pi : \mathcal{X} \to X$

- GMS are topol. mod spaces, defined by

(1) π universally closed & $\forall x \in X$
$\exists ! \; x_0 \in \pi^{-1}(x)$ closed

(2) $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$

(3) $\forall \; \mathcal{X}' \xrightarrow{\text{finite}} \mathcal{X}$, $\mathcal{X}' \to \operatorname{Spec}(\pi_* \mathcal{O}_{\mathcal{X}'})$
should satisfy (1)

Non-example a) $[\mathbb{P}^1/\mathbb{G}_m] \to \circ$ violates (1)

b) $[\alpha / \mathbb{G}_m]$ satisfies (1) but not (3)

Thm: If $\pi : \mathcal{X} \to X$ top. mod. sp.

(1) π is universal (initial among maps to alg. spaces)

(2) $\mathcal{X}$ finite type $\Rightarrow X$ finite type

(3) π satisfies Luna's fund. theorem

Local str. of gms

Thm AHR '19 "gms $\Leftrightarrow$ étale-locally affine GIT"

If $\pi : \mathcal{X} \to X$ gms $\exists X' \to X$ étale and $G \curvearrowright \operatorname{Spec} A$

$$\begin{tikzcd} {[\operatorname{Spec} A / G] = \mathcal{X} \times_X X'} \arrow[r] \arrow[d] & \mathcal{X} \ni x_0 \arrow[d] \\ \operatorname{Spec} A^G = X' \arrow[r] & X \ni x \end{tikzcd}$$

G red
$\shortparallel$
$\operatorname{Stab}(x_0)$

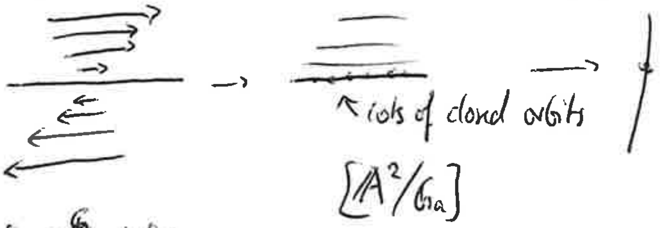
- Local str. around $x \in \mathcal{X}$ with red. stab w/o $\mathcal{X}$ having gms
- Canonical stabilizer red. by blowups (Kirwan partial desing., Edidin-R)
- Existence crit. for having gms (Alper-Halpern-Leistner-Heinloth '17)

David Rydh

§ Non-reductive

H any linear group $\quad 1 \to \overset{\text{unipotent}}{U} \to H \xrightarrow{} \underset{\text{reductive}}{R} \to 1$ $\quad$ Mostow char 0!

Example $\mathbb{G}_a \circlearrowright \mathbb{A}^2 \quad t(x,y) = (x+ty, y)$



$k[x,y]^{\mathbb{G}_a} = k[y]$

very bad

- $\not\exists$ closed point of $\pi^{-1}(0)$
- not comm. w base change
  $\left(k[x,y]/(y)\right)^{\mathbb{G}_a} = k[x]$
- In general A^H not f.gen.

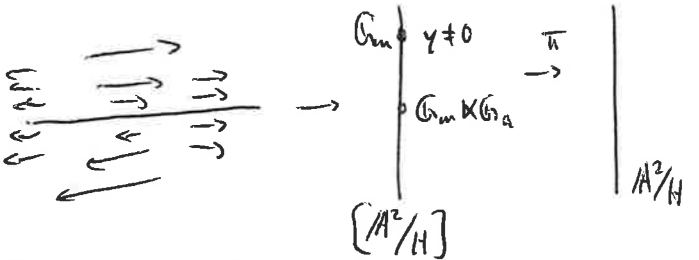
Ex: $\mathbb{G}_m \ltimes \mathbb{G}_a \curvearrowright \mathbb{A}^2 \quad t.(x,y) = (x+ty, y)$

$\lambda, t \qquad \lambda.(\lambda^{w_1} x_1, \lambda^{w_2} y)$

$\mathbb{G}_m \curvearrowright \mathbb{G}_a$ weight $w_1 - w_2 = d$

Claim: well behaved if $w_1 > w_2 \geq 0$

Ex: $w_1 = 1, w_2 = 0$



Reason: $k[x,y]^H = k[x,y]^R \Rightarrow$ f.gen. etc

Lemma: $V \in \mathrm{Rep}(H)$ then $V^H = V^R$ iff

$\forall V \twoheadrightarrow W \Rightarrow V^H \twoheadrightarrow W^H$

Def: $\pi : \mathcal{X} \to X$ is a NR_0 gms if $\quad$ "0-good NR gms"

1) $\pi_* \mathcal{O}_{\mathcal{X}} \twoheadrightarrow \pi_* \mathcal{F}$ surjective $\quad \forall \mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{F}$

2) $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$

3) π is a top. mod space

Ex: $\mathcal{X} = [\operatorname{Spec} A/H] \to \operatorname{Spec} A^H = X$ is NR_0-gms

if $A^H = A^R$

Ex: $\mathcal{W} \xrightarrow{p} \mathcal{X} \xrightarrow{\pi} X$ st.

a) p smooth with geom. conn. fibers

b) $\pi \circ p$ is a gms

$\Rightarrow \pi$ is a NR_0 gms

Rmk: Ex 2 $\Rightarrow$ Ex 1

$[\operatorname{Spec} A/R] \xrightarrow[p]{U\text{-fibration}} [\operatorname{Spec} A/H] \to \operatorname{Spec} A^H$

Results:

Thm 1: NR_0 gms $\quad$ (local str. thm)

$\pi : \mathcal{X} \to X$

$\Updownarrow$

etale locally of

$$\begin{tikzcd}
\mathcal{W}' \arrow[dr] \arrow[ddr, bend right, "\text{gms}"'] & & \\
& \mathcal{X}' \arrow[r] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\
& X' \arrow[r] & X
\end{tikzcd}$$

$\exists$ NR_d gms $\qquad NR_+$ gms

Kirwan's setup $\Rightarrow NR_+$ gms

Thm 2: can. stab. red

$\exists$ can. seq. of blow-ups that red. stab. dim.

Thm 3: $\pi : \mathcal{X} \to X$ top mod. sp.

a) gms if all stab(x_0) red. $\forall x_0 \in \pi^{-1}(x)$

b) NR_d gms $\quad$ stab(x_0) $\curvearrowright T_{x_0}\mathcal{X}$ is d-good — u —

Def: $H \curvearrowright V$ is d-good if $\quad d \geq 1$

$(V \otimes \mathcal{O}_u^{\otimes d})^R = V^H$

Rmk: V = k triv. rep

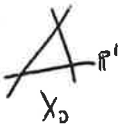
V is d-good $\Leftrightarrow U^d /\!/ R = *$

Vivek Shende: Towards HMS

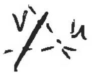
Let X_0 be a union of toric varieties
then ∃ noncpt symp. mfd $X_0^\vee$ s.t.
$Coh(X_0) = Fuk(X_0^\vee)$

Moreover for any open toric subset $U \subseteq X_0$, $V = X_0 \setminus U$

$$\begin{tikzcd}
Coh(V) \arrow[r] \arrow[d, equal] & Coh(X_0) \arrow[r] \arrow[d, equal] & Coh(U) \arrow[d, equal] \\
Fuk(V^\vee) \arrow[r] & Fuk(X_0^\vee) \arrow[r] & Fuk(U^\vee)
\end{tikzcd}$$

Ex 

⟷  $U^\vee$



Obvious $Def(Coh X_0) = Def(Fuk(X_0^\vee))$

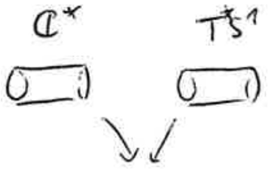
given a smooth X_t of X_0

$$\begin{tikzcd}
Coh(X_t) \arrow[r, phantom, "\overset{?}{=}"' , "match"] & Fuk(X_0^\vee) \arrow[d, squiggly] \\
Coh(X_0) \arrow[u, squiggly] & Fuk(X^\vee)
\end{tikzcd}$$

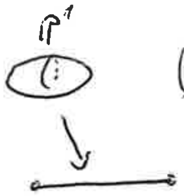
$X_0^\vee \hookrightarrow \overline{X_0^\vee}$ ample divisor complement

$$\begin{tikzcd}
Def(Coh X_0) & = H^0 Sym T_X[1] [[\varepsilon]] \\
Def(X_0) \arrow[u] & = H^0 T_{X_0}[1] [[\ldots]] \arrow[u, "?"] \\
Def(X_{0,log}) \arrow[u] & = H^0 T_{X_{0,log}}[1] [[\ldots]] \arrow[u]
\end{tikzcd}$$

$\varepsilon - 1?$



$Coh(\mathbb{C}^*) = Fuk(T^*S^1)$

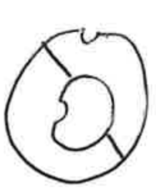


circle of ∞ radius

mirror of $\mathbb{P}^1$

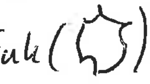
displacement yields zero # points

Hom in Fuk in noncompact case requires you to flow one of the Lag by Reeb flow



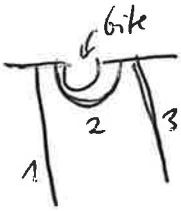
two copies of 

Thm
Ganatra, Pardon-S
Fuk commutes with sectorial twists

$Fuk($  $) = Rep(\bullet \to \circ)$

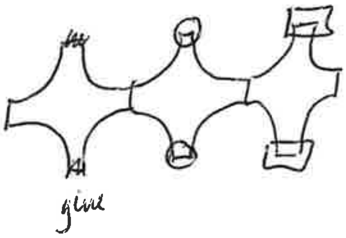
Reeb flow makes $Hom(L,L') \neq Hom(L',L)$

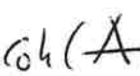
$Rep\left(\begin{smallmatrix} \bullet & \to & \circ \\ \| & & \| \\ \circ & \to & \circ \end{smallmatrix}\right)$ = dg-cat of $\mathbb{P}^1$



$Cone(1,2) = 3$

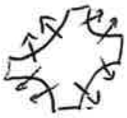
Reeb flow must stop at the bite
so $Coh(\mathbb{C}^*)$ is a localization of $Coh(\mathbb{P}^1)$ by 

Fuk 

$= Coh($  $)$

Lag skeleta W exact symp mfd
$\omega = d\lambda$ $\omega(Z, \cdot) = \lambda(\cdot)$

"Liouville"
locus of what doesn't escape is a singular Lagrangian.

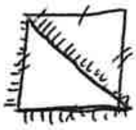
 skeleton is 

Thm $Fuk = \Gamma$(microlocal sheaf on skeleton)

Def: (Feng-Liu-Treumann-Zaslow 2010)
Given $\Sigma \subset N_{\mathbb{R}}$ $\Lambda = \bigcup_{\sigma \in \Sigma} \underset{\underset{N_{\mathbb{R}}}{\uparrow}}{\sigma} \times \underset{\underset{M_{\mathbb{R}}/M}{\uparrow}}{\sigma^\perp} \subseteq T^*(M_{\mathbb{R}}/M)$

Example:
$\mathbb{P}^1 \longleftrightarrow$

$- M_{\mathbb{R}}/M$



Twisted FM partners of ordinary K3

Sofia
Tirabassi

Surfaces jw T. Srivestava

Goal: ① "finiteness" of twisted FM partner of K3 in char p.

② For ordinary K3s the relation between twisted FM and canonical lift.

§ Twisted sheaves

X smooth projective$/k=\bar{k}$

$Br'(X) := H^2_{\acute{e}t}(X, \mathcal{O}_X^*)_{tor}$

$\alpha \in Br'(X) \rightsquigarrow Coh(X,\alpha) \quad D^b(X,\alpha)$

a) Căldăraru's way:
use α to twist the gluing condition of sheaves on étale cover Fix $\{U_i\}$

$Coh(X,\alpha) \ni \mathcal{F} \leftrightarrow \{\mathcal{F}_i, \alpha\varphi_{ij}\}$

b) Bernadara/Yoshioka way
α has order n $(n, \mathrm{char}\, k) = 1$

$$Pic(X)/_{n}Pic(X) \to H^2_{\acute{e}t}(X, \mu_n) \twoheadrightarrow Br'(X)[n]$$

$H^1(X, PGL_n) \ni [P_a]$

You can view twisted sheaves as some sort of eigensheaves on $P_a \to X$

$$\begin{tikzcd} P_a \arrow[r] & X \\ \mathcal{O}(1) \quad \mathbb{P} \arrow[u] \arrow[r] & U\, p^*\mathcal{F}\otimes\mathcal{O}(1) \arrow[u] \end{tikzcd}$$

③ The right way

$\mathcal{X}_{\mathbb{G}_m} \to X \qquad \mathbb{G}_m$ gerbe

$[\mathcal{X}_{\mathbb{G}_m}] = \alpha$

$$D^b(\mathcal{X}_{\mathbb{G}_m}) \simeq \prod_{n\in\mathbb{Z}} D^b(\mathcal{X}_{\mathbb{G}_m}, n)$$

$$D^b(X,\alpha) \simeq D^b(\mathcal{X}_{\mathbb{G}_m}, 1)$$

Twisted FM partners of ordinary K3 surfaces

jw Srivestava

We say that two varieties X and Y are twisted Fourier-Mukai partners if there is an exact equivalence

$D^b(X,\alpha) \to D^b(Y,\beta)$

for some $\alpha \in Br'(X)$, $\beta \in Br'(Y)$

Twisted FM and canonical lift

$FM(X,\alpha) = \{(Y,\beta)$ such that $D^b(Y,\beta) \simeq D^b(X,\alpha)$

Ma: X K3$/\mathbb{C}$

$\# FM^d(X,1)$ - Hodge theoretic formula

$d \,/\,$ order of β

§ K3 in char p

X smooth proj. surface $K_X \sim 0$

$h^1(X, \mathcal{O}_X) = 0$

RUDAKOV-SHAFAREVICH $H^2(X, \mathcal{O}_X) = 0$

$\Rightarrow$ Given R a complete local W-algebra then there is a complete formal lift $\mathcal{X} \downarrow R$ $\quad \mathcal{X}/_k = X$

Deligne $\exists$ proj. lift

The formal Brauer group

$$\hat{Br}_X : (Art/k) \longrightarrow (Ab)$$
$$A \longmapsto \ker(Br(X_A) \to Br(X))$$

$\qquad\qquad \overset{\shortparallel}{\{\text{Azumaya algebras}/X_A\}}/_{\text{Morita}}$

Artin-Mazur

This is pro-rep by a formal group $\hat{Br}(X)$ the formal Brauer gp

The height of $\hat{Br}(X)$ is denoted by $ht(X)$ and is called the height of X.

If $ht(X) = +\infty$, X is supersingular

Bragg-Lieblich.

If $ht(X) < \infty$ you can show that there is a Picard preserving lift $\mathcal{X} \downarrow \mathrm{Spec} R$ $\quad \mathcal{X}_k \cong X$

the specializing map $Pic(\mathcal{X}) \downarrow Pic(X)$

$\phi : D^b(X,\alpha) \to D^b(Y,\beta)$

$\exists\, P \in D^b(X\times Y, p^*(\alpha)^{-1} \otimes q^*(\beta))$

$\phi = \phi_P$ it descends in cohomology

$\phi^H(0,0,1) = ?$ $\quad v = (r, 2, x)$

$M_{(X,\alpha)}(v) \simeq Y$

$h(X) = 1 \iff X$ is ordinary $\quad F^0H(X,\mathcal{O}_X) \xrightarrow{\sim} H^0(X,\mathcal{O}_X)$

R complete W-algebra $\mathcal{X} \to R$ formal

$$\Psi_{\mathcal{X}_A} : (Art/R) \to (Ab)$$
$$B \longmapsto H^2_{fppf}(\mathcal{X}_B, \mu_{p^\infty})$$

Artin-Mazur $\Psi(X_A) \longrightarrow \mathrm{Spec}(A)$ p-div-gr

$$0 \to \Psi_X^{\circ} \hookrightarrow \Psi(X) \to \Psi^{et}(X) \to 0$$

$$\mathrm{Def}_X(A) \longrightarrow \mathrm{Ext}^1(\Psi^{et}(X), \widehat{Br}(X))$$

Nygard : the canonial lift is projective

How to make log structures

Alessio Corti

Historical motivation
$$\begin{tikzcd} Y \arrow[r, phantom, "\subset"] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r, phantom, "\subset"] & T = \operatorname{Spec} \mathbb{C}[t] \end{tikzcd}$$
semistable

$\forall y \in Y \ \exists z_1 \dots z_n$

$f = z_1 \cdot \dots \cdot z_r$ $\qquad \Omega^1_X \langle \log Y \rangle$

$$0 \to \Omega^1_T \langle 0 \rangle \to \Omega^1_X \langle \log Y \rangle \to \underbrace{\Omega^1_{X/T} \langle \log Y \rangle}_{\text{locally free}} \to 0$$

Def: A pre-log structure (M, α) on Y

M : sheaf of monoids

$\alpha : (M, +) \to (\mathcal{O}_X, \cdot)$

is a log str. if $\alpha^{-1}(\mathcal{O}_Y^\times) \xrightarrow{\alpha} \mathcal{O}_Y^\times$ is an iso

$\underbrace{\frac{dz_1}{z_1} \dots \frac{dz_r}{z_r}}_{d\log z_i}, dz_{r+1}, \dots dz_n$

M = things of which we can make "dlog"

Example $Y \subset X$ $\quad \mathcal{M}_{(X,Y)}(U) = \mathcal{O}(U \setminus Y)^\times \cap \mathcal{O}_X(U)$

Ghost sheaf

$\mathcal{M} / \mathcal{O}_X^\times = \overline{P}$

Morphism
$$\begin{tikzcd} \mathcal{M} \arrow[rr, dashed] \arrow[dr, "\alpha"'] & & \mathcal{M}' \arrow[dl, "\alpha'"] \\ & \mathcal{O}_X & \end{tikzcd}$$

to a pre-log str. there is a canonically associated log structure.

$\beta : N \to \mathcal{O}_X$ $\rightsquigarrow$
pre-log
$$\begin{tikzcd} \beta^{-1}\mathcal{O}_X^\times \arrow[r] \arrow[d] & \mathcal{O}_X^\times \arrow[d] \\ N \arrow[r] & N^a \end{tikzcd}$$

$\operatorname{Mor}_{\log}(N^a, \mathcal{M}) = \operatorname{Mor}_{\text{pre}}(N, \mathcal{M})$

Example $Y = \{ z_1 \cdot \dots \cdot z_r = 0 \} \subset X = \mathbb{C}^n$

$\beta : \mathbb{N}^r \to \mathcal{O}_X$

$(n_1 \dots n_r) \mapsto z_1^{n_1} \cdot \dots \cdot z_r^{n_r}$

$$(\mathbb{N}^r)^a = \mathcal{M}_{(X,Y)} = \coprod_{(n_1 \dots n_r)} z^{n_1} \cdot \dots \cdot z^{n_r} \cdot \mathcal{O}_X^\times$$

If $f : X \to Y$ log str. can be pushed fwd pulled back etc.

$X^\dagger = (X, N)$

$Y^\dagger (Y, M)$ $\qquad$ morphism of log schemes

$f^* M \to N$

$f^{-1} M \to f^{-1} \mathcal{O}_Y \to \mathcal{O}_X$

The log structures we are interested in

$Y = (z_1 \cdot \dots \cdot z_r = 0)$ $\qquad \mathcal{M} = i^* M_{(X,Y)} \underset{\text{locally}}{=\!=} \coprod_{(n_1 \dots n_r)} z_1^{n_1} \cdot \dots \cdot z_r^{n_r} \mathcal{O}_Y^\times$

More generally

$Y \underset{\text{locally}}{=} \operatorname{Spec} k[\Sigma^\vee]$ $\qquad \mathcal{M} = \coprod_{m \in |\Sigma| \cap M} x^m \mathcal{O}_Y^\times$

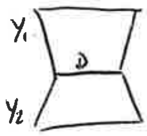
Stanley-Reisner ring

$\exists Z \subseteq Y$ codim $Z \geq 2$ so that $Y \setminus Z$ has this form

Our motivation

- Smooth Y to make examples of alg. varieties (Fano varieties)
- Log birational geometry

We want to have practical methods to construct a log structure on Y



(1) to give a log str on Y: $\mathcal{L}_1, \mathcal{L}_2$ on Y

$\alpha_1 : \mathcal{L}_1 \to \mathcal{O}_Y$ $\qquad \mathcal{L}_1|_{Y_2} = \mathcal{O}_{Y_2}(-D)$

$\alpha_2 : \mathcal{L}_2 \to \mathcal{O}_Y$ $\qquad \mathcal{L}_2|_{Y_1} = \mathcal{O}_{Y_1}(-D)$

$$\mathcal{M}(U) = \coprod_{n_1, n_2 \in \mathbb{N}^2} (\mathcal{L}_1^\times)^{n_1} \otimes (\mathcal{L}_2^\times)^{n_2}$$

$U \cap D \neq 0$

if $U \subset Y_1 \setminus D$ $\quad \mathcal{M}(U) = \coprod_{(n_1, 0)} (\mathcal{L}_1^\times)^{n_1}$

Also want morphism to $(\operatorname{Spec} k)^\dagger = \operatorname{Spec}(\mathbb{N} \to k)$ $\quad \begin{matrix} 1 \mapsto 0 \\ 0 \mapsto 1 \end{matrix}$ $\;" \overset{i}{\hookrightarrow} \mathbb{A}^1_k$

(2) to give log str. on Y + morphism to $(\operatorname{Spec} k)^\dagger$ $\qquad i^* \mathcal{M}_{(\mathbb{A}^1, 0)}$

$s \in H^0(D, N_D Y_1 \otimes N_D Y_2)$ never vanishing

$$\begin{tikzcd} Y^\dagger \arrow[r] & (\operatorname{Spec} k)^\dagger \\ \mathcal{M} \arrow[d] & \mathbb{N} \arrow[l] \\ \mathbb{1} & 1 \arrow[l, maps to] \end{tikzcd}$$

and 1 trivializes $\mathcal{L}_1 \otimes \mathcal{L}_2$

earlier: (in (1)) $\mathcal{L}_1|_{Y_2} \simeq \mathcal{O}_{Y_2}(-D)$

now: (in (2)) $\mathcal{L}_1|_{Y_2} \simeq \mathcal{L}_2^*|_{Y_1} = \mathcal{O}_{Y_1}(D)$

$N_D Y_1 \simeq N_D Y_2^*$

↑ this isomorphis is the data of

$s \in H^0(D, N_D Y_1 \otimes N_D Y_2)$
non-vanishing

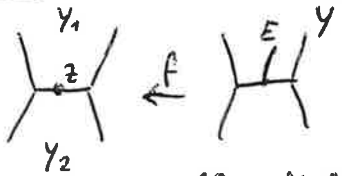
Thm:

Y generic toroidal crossing (+assumption) then

$\exists$ sheaf $\mathcal{LS}_Y^*$ intrinsic to Y, explicitly computable so that

$$H^0(Y, \mathcal{LS}_Y^*) = \left\{ \begin{matrix} \text{log str} \\ \text{on } Y \end{matrix} \to \operatorname{Spec} k^\dagger \right] \Big/ \text{iso}$$

Possible



$$\mathcal{LS}_{\tilde{Y}} = f^* \mathcal{LS}_Y(-Z)$$

Vivek's question: is $\mathcal{LS}_Y$ known from abstract theory

Answer: $\mathcal{LS}_Y^*$ is known but not $\mathcal{LS}_Y$.

Lutz Hille

Polynomial invariants for full except. sequences

§1 motivation

$\mathcal{T} = \mathcal{D}^b \mathrm{Coh}\, X \quad \mathcal{D}^b(\mathrm{mod} A)$

X proj smooth

$\mathcal{E} = (E_1 \dots E_n)$

A f.d. algebra (quasi-hered.)

$\Delta = (\Delta_1 .. \Delta_n)$ standard modules

$\mathcal{T} \simeq \mathcal{T}'$?

$(\mathcal{T}, \varepsilon) \simeq (\mathcal{T}', \varepsilon')$?

triang. full exc. $\bigoplus_{j<i} P_j \to P_i \twoheadrightarrow \Delta_i$

1) $(K_0(\mathcal{T}), \langle , \rangle) \simeq (K_0(\mathcal{T}'), \langle , \rangle')$ $\quad K_0(\mathcal{T}) \simeq \mathbb{Z}^n$

2) $\mathcal{T} \simeq \mathcal{T}'$ underived

Def: $\mathcal{E} = (E_1 .. E_n)$ full $\langle E_1 .. E_n \rangle \simeq \mathcal{T}$ $\quad k = \bar{k}$

E_i exceptional if $\mathrm{End}(E_i) = k$ $\quad \mathrm{Ext}^q(E_i, E_i) = 0$

$\mathcal{E}$ ex. seq if E_i exc and $\mathrm{Ext}^q(E_j, E_i) = 0$ $\quad q \neq 0$ $\forall q$ $\forall j > i$

$\chi(E_i, E_j) = \sum'_{q \in \mathbb{Z}} (-1)^q \dim \mathrm{Ext}^q(E_i, E_j) \in \mathbb{Z}$

$C(\mathcal{E}) = \begin{pmatrix} 1 & & \chi(E_i, E_j) \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in M_n(R_n)$

$R_n = \mathbb{Z}[x_{ij} \mid 1 \le i < j \le n]$

$\simeq \mathbb{Z}[x_{ij} \mid 1 \le i, j \le n] \,/\, (x_{ij} = x_{ji},\ x_{ii} = 1)$

Examples for X

$\mathbb{P}^n$ Beilinson

$\mathbb{F} = Gl_n/P$ Kapranov flag mfd

Thm: X rational surface then X admits a full exc. seq. of line bdls $(L_1 .. L_n)$

$\mathbb{F}_n$ Hirzebruch, explicitly from line bundles

$\tilde{X} \to X \qquad \mathcal{E} \rightsquigarrow \tilde{\mathcal{E}} = (L_1(E) \dots L_{i-1}(E), L_i, L_i(E) L_{i+1} \dots L_n)$

full exc. on $\tilde{X}$

$F \in R_n$ a polynomial

$F(\mathcal{E}) = F(\chi(E_i, E_j)) \qquad x_{ij} = \chi(E_i, E_j)$

Do $C(\varepsilon)$ satisfy relations?

Action $\mathcal{C}(\mathcal{T})$ set of full exc. seq.

1) $(E_1 .. E_n) \xrightarrow{T_i} (E_1 \dots E_i[1] \dots E_n)$ $\curvearrowleft \mathbb{Z}^n \rightsquigarrow (\mathbb{Z}/2)^n$ on R_n

2) $(E_1 .. E_n) \overset{L_i}{\longmapsto} (E_1 .. L_{E_i} E_{i+1}\, E_i\, E_{i+2} \dots)$ $\quad x_{ij} \mapsto \begin{cases} -x_{ij} & i = j \\ x_{ij} & \text{else} \end{cases}$

induces braid group action

B_n acts on R_n $\quad \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} \longmapsto \begin{pmatrix} \triangledown & & \square \\ & 1\, x_{i+1} & \\ & & \triangledown \end{pmatrix}$

$L_{E_i} E_{i+1} \to \bigoplus_q \mathrm{Hom}(E_i, E_{i+1}[q]) \otimes E_i[-q] \to E_{i+1} \to L[1]$

Question Describe: $R_n^{B_n} \supsetneq B_n^{B_n, \mathbb{Z}/2^n}$? $\qquad n \le 5$ explicit

3) $(E_1 \dots E_n) \mapsto (L^{n-1} E_n, E_1 \dots E_{n-1})$ $\quad \mathbb{Z}/n$-action $\subseteq B_n$

$R_n^{B_n} = R_n^{G, \mathbb{Z}_n}$

Rel $(\mathcal{T}, \varepsilon)$ $\quad$ F is a polynomial invariant if $F(\varepsilon)$ does not depend on choice of ε

n fixed

$F(\mathcal{T}) = F(\mathcal{E})$ invariant of $\mathcal{T}$

Example $\qquad (R_n^{B_n} \supseteq R_n^{pol})$ factors on $K_0(\mathcal{T}), \langle , \rangle$

$\mathcal{E} = (E_1 .. E_n)$ full exc on $\mathbb{P}^2$

$r_1^2 + r_2^2 + r_3^2 - 3 r_1 r_2 r_3 = 0$ Markov equ

$\Leftrightarrow F_1 = x_{12}^2 + x_{13}^2 + x_{23}^2 - x_{12} x_{23} x_{13} = C$ $\quad$ for any ε $\quad n = 3$

$F_1 \in R_3^{B_3}, R^{pol}$

$\mathrm{Tr}(C^t C^{-1}) \quad \begin{pmatrix} 1 & & \\ x & 1 & \\ y & z & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & xz - y \\ & 1 & -z \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x^2 + 1 & \\ & & -y^2 - z^2 + xyz + 1 \end{pmatrix}$

$\phi = C^t C^{-1} \simeq$ Serre functor on K_0

$\mathcal{E} \rightsquigarrow \mathcal{E}'$ base locus in K_0 $\quad \phi \rightsquigarrow \phi'$ complicated

Construction

$\det(t C^t + C)$

$= \det(t\, \mathrm{id} + C C^{-t})$

$= \det(t\, \mathrm{id} + C^t C^{-1})$

$= \sum_{i=0}^{n} F_i t^i \qquad F_i = F_{n-i}$ by symmetry

$\underline{n=3}$ $R_3^{B_3} = R_3^{pol} = \mathbb{Z}[F_1]$

$\underline{n=4}$ $F_1 = \sum' x_{ij}^2 - \sum' x_{ij} x_{jk} x_{ik} + x_{12} x_{23} x_{34} x_{14}$

$R_4^{Br} \ni \sqrt{G_2} = x_{12} x_{34} - x_{13} x_{24} + x_{14} x_{23}$

$R_n^{pol} \qquad G_n \in R_n^{pol}$

get $F_1, \dots F_{\lfloor n/2 \rfloor}$ pol. inv.

$\boxed{\text{Thm:}}$ these generated R_n^{pol} freely

Conj $R_n^{Br} = R_n^{pol}$ if n odd

$\dots [\sqrt{G_{\frac{n}{2}}}]$ if n even $\qquad$ proved for $n \le 5$

$\sqrt{G_{n/2}} = \mathrm{Pf}(C^t - C) \quad t = -1$

§3 Main result

$\boxed{\text{Thm}}$ a) $F_i(\mathcal{T}) \in \mathbb{Z} \Leftrightarrow$ b) eigenvalues of ϕ

c) $(K_0, \langle , \rangle) = (K_0', \langle , \rangle)' \Leftrightarrow$ d) $\phi \sim \phi'$ conj.

Derived equivalence of IHS manifolds

over $\mathbb{C}$

Michal Kapustka
j.w.
Kapustka

X - such that K_X, $-K_X$ ample
determines X up to isomorphism

$K_X = 0$ this is not the case

X, Y $\quad D^b(X) \simeq D^b(Y)$ $\quad X, Y$ der. equiv. of

FM partners for CY type surfaces

1) abelian surfaces

$A, A^\vee$ Mukai $D^b(A) \simeq D^b(A^\vee)$

$A \times A^\vee$ Poincaré sheaf

$F \mapsto \pi_{2*}(P \otimes \pi_1^* F)$

2) K3 surfaces

Thm Orlov $\quad S_1, S_2$ are derived equivalent

$\iff Tr(S_1) \simeq Tr(S_2)$

isometry of integral Hodge

$NS(S) = \langle 2d \rangle$

FM partners $= 2$

Example $\quad S$ K3 surface of degree 20 $\quad \rho(S) = 1$

$$f(x) = x - \frac{2(x,v)}{(x,x)} v$$

$$H^2(S, \mathbb{Z}) = 2H \oplus 2E_8(-1)$$

Def: X IHS if sim. simply conn. mfd $H^0(X, \Omega^2) = \mathbb{C}\omega_X$

Several known
- $K3^{[n]}$ type defo of Hilb of points on K3
- gen. Kummer
- $OG6$, $OG10$

Today: $K3^{[n]}$

aim: Orlov for $K3^{[n]}$ type IHS mfd

Examples of FM partners

1) $D^b(S_1) = D^b(S_2)$ are equivalent

$D^b(S_1^{[n]}) = D^b(S_2^{[n]})$ $\quad$ Ploog

2) Moduli of torsion sheaves on K3 surfaces

S $\quad (0, H, \alpha)$

$M_{(0,H,\alpha)}(S) \underset{\substack{\text{twisted} \\ \text{derived} \\ \text{equiv.}}}{\simeq} M_{(0,H,\beta)}(S)$ $\quad$ we can identify twists

abelian fibrations $\quad$ dual abelian fibrations

relative Poincaré twisted sheaf $\quad$ Addington, Donovan

$D^b(X) \simeq D^b(Y) \underset{\text{Taelman}}{\rightsquigarrow} H^2(X, \mathbb{Q}) \simeq H^2(Y, \mathbb{Q})$

Restrictions to Hodge isometries on integral transc. lattices

idea: take known examples and deform the kernel

aim: prove $\Leftarrow$ for X, Y IHS $\rho(X) = \rho(Y) = 1$

Working example

EPW sextics $\quad W_6 = 6$-dim^l vect.sp.

$\Lambda^3 W_6$ skew form given by wedge product

$X_A \xrightarrow{2:1} \overline{X}_A = \{ v : F_v \cap \mathbb{P}(A) \neq \phi \}$

sextic hypersurface

$A \subset \Lambda^3 W_6$

$\mathbb{P}(A) \subset \mathbb{P}(\Lambda^3 W_6) \supset G(3, W_6)$

$$\begin{tikzcd} \overline{X}_A^\vee = \{ v : F_v^* \cap \mathbb{P}(A) \neq \phi \} \\ X_A^\vee \arrow[u, "2:1"] \end{tikzcd}$$

dual double EPW sextic

For $v \in \mathbb{P}(W_6)$ $\quad G(v, 3, W_6) \simeq G(2,5)$ spans $\mathbb{P}^9$

also Lagr. $F_v : v \in \mathbb{P}(W_6)$

$X_5 \subset \mathbb{P}(W_6)$ $\quad G(3, V_5) \simeq G(2,5)$

spans $\mathbb{P}^9$ Lagrangian

$\{ F^*_{V_5} : V_5 \in \mathbb{P}(W_6))$

X_A of K3 type

$H^2(X, \mathbb{Z}) = 3U \oplus 2E_8(-1) \oplus \langle -2 \rangle$

$NS(X_A) = \langle \underset{2}{e+f} \rangle$

$Tr(X_A) = \langle \underset{\textcircled{-2}}{e-f} \rangle \oplus 2U \oplus 2E_8(-1) \oplus \underset{\delta}{\langle -2 \rangle}$

δ reflection by -2 class with divisibility 1

Cohomological rank functions on Abelian Surfaces via Bridgeland Stability

Martí Lahoz

(based on j.w. Andres Rojas and work by himself)

SYZYGIES

Properties (N_p) $\qquad (X, L)$ ← proj var, ← very ample

$$R_L = \bigoplus_m H^0(X, L^m) \qquad S = \mathrm{Sym}\, H^0(X, L)$$

$(N_0) \Leftrightarrow R_L \xrightarrow{\varphi_0} S$ proj. normal

$(N_1) \Leftrightarrow (N_0) + \ker \varphi_0 = I_{X/\mathbb{P}(H^0(X,L)^\vee)}$ generated by quadrics

$$S(-2)^{\oplus a_i} \xrightarrow[\varphi_1]{} I_{X/P}$$

Green curves

(C, L) (N_0) is satisfied if deg is big enough

Conj (C, ω_C) satisfied $(N_p) \Leftrightarrow \mathrm{Cliff}(C) > p$

Proven by Voisin for generic curves ('02)

Abelian varieties

Lazarsfeld conj (A, L) pol. abelian variety, L ample

L^m satisfies N_p if $m \geq p+3$

prove for char 0 in 2000 Pareschi

char p in 2020 Caucci

cohomological rank function CRF

§ CRF & N_p for abelian var

(A, L) pola. abel. $g = \dim A$

$\mu_b : A \to A$ multiplication by b

$D^b(A) = D^b(\mathrm{Coh}\, A)$

Def Jiang Pareschi 2020) $F \in D^b(A)$ $i \in \mathbb{Z}$

$x \in \mathbb{Q}$ $\quad h^i_{F,L}(x)$ "=" $h^i(F \otimes L^x \otimes \alpha)$ $\qquad \alpha \in \mathrm{Pic}^0(A)$ general

$x = \frac{a}{b} \qquad = \frac{1}{b^{2g}} h^i(\mu_b^* F \otimes L^{ab} \otimes \alpha)$

$$h^i_{F,L} : \mathbb{Q} \to \mathbb{Q}_{\geq 0}$$

Example (A, L) $\quad \dim A = g$

$0 \in A$ $\quad F = I_0$ $\quad h^0(L) = d$

$h^i_{F,L}$ extends to continuous $\mathbb{R} \to \mathbb{R}_{\geq 0}$

$$h^0_{I_0,L}(x) = \begin{cases} 0 & x \leq 0 \\ ? & \sim \\ dx^g - 1 & x \geq 1 \end{cases}$$

$\beta(L) = \sup\{x \mid h^0_{F,L}(x) = 0\}$ bpf threshold

JP: $\beta(L) = 1 \Leftrightarrow L$ has basepoints

$\beta(L) < \frac{1}{2} \Leftrightarrow L$ is proj normal

Caucci $\boxed{\beta(L) < \frac{1}{p+2}} \Leftrightarrow L$ satisfies (N_p)

→ prove Lazarsfeld in char p

Ito $\beta(L) \leq \frac{1}{p+2}$ & h^0 is C^1 at $\frac{1}{p+2}$

$\Rightarrow L$ satisfies (N_p)

Ito (A, L) ← surface $\quad (1, d)$ $\quad \beta(L) \leq \frac{1}{\lfloor \sqrt{d} \rfloor}$ and if $m = \lfloor \sqrt{d} \rfloor$, $d \geq m^2 + m + 1$

$$\beta(L) \leq \frac{\lfloor \sqrt{d} \rfloor + 1}{d}$$

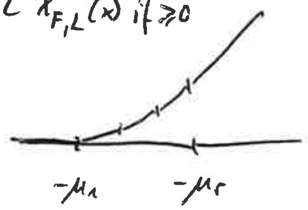
Idea: For surface (A, L)

Use stability to compute CRF

Example: Elliptic curve $F \in \mathrm{Coh}(E)$ $\quad L = \mathcal{O}(pt)$

F is μ-sst $\quad h^0_{F,L} = \begin{cases} 0 \\ \chi_{F,L}(x) \text{ if } \geq 0 \end{cases}$

$\mu_1 > \ldots > \mu_r$ ← slopes of Harder-N. filtration



(A, L) abelian surface

$NS(A) = \mathbb{Z} \cdot L$ $\qquad$ Stab = (α,β) plane ← geometric stability

$\forall (\alpha, \beta) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \hookrightarrow \mathrm{Stab}(A)$

$$\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(A), \nu_{\alpha,\beta})$$

$\mathrm{Coh}^{\beta^2}(A)$ = tilt of $\mathrm{Coh}(X)$ at $\mu = \beta$

$$\nu_{\alpha,\beta}(E) = \frac{ch_2^{\beta^2}(E) - \frac{\alpha^2}{2} ch_0(E)}{L \cdot ch_1^{\beta^2}(E)}$$

$\mathrm{Coh}^\beta(X) \subseteq D^b(X)$ heart

Thm (L-Rojas) $x \in \mathbb{Q}$ $\qquad$ analogy w. elliptic curves

a) If $F \in \mathrm{Coh}^{-x}(X)$ then $h^i_{F,L}(x) = 0$ if $i \neq 0, 1$

b) If $0 = F_0 \hookrightarrow F_1 \hookrightarrow F_s \hookrightarrow F_{s+1} \hookrightarrow F_r = F$ HN filt wrt $\sigma_{0,-x}$

$V_{0,-x}\left(F_{s+1}/F_s\right) \geq 0$

$\sigma_{0,\beta}$ weak stability condition (Arena-Bertram)

$\mathrm{Coh}^{\beta}(X)$ heart of
$\cap$
$D^b(X)$

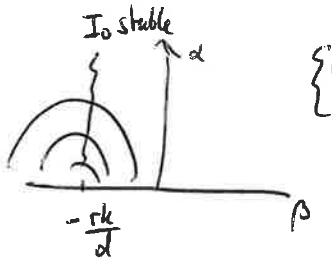
$h^0_{F,L}(x) = \mathrm{ch}_2^{-x}(F_s)$

$h^1_{F,L}(x) = -\mathrm{ch}_2^{-x}(F/F_s)$

c) Formula for any $F \in D^b(X)$

Application to syzygies (Rojas) (X,L) pol. ab surface

$F = I_0 \in \mathrm{Coh}^{\beta}(X) \Leftrightarrow \beta < 0$ I_0 is $\sigma_{\alpha,\beta}$-stable if $\alpha \gg 0$



$\left\{\begin{matrix}\text{possible walls} \\ \text{for } I_0\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\text{pos sol. to} \\ \text{the Pell eqn} \\ x^2-4dy^2=1\end{matrix}\right\}$

<u>Thm</u> Rojas $NS(X) = \mathbb{Z}\cdot L$

(1) d is a square
$$h^0_{I_0,L}(x) = \begin{cases} 0 & x \leq \frac{\sqrt{d}}{d} \\ dx^2-1 & x \geq \frac{\sqrt{d}}{d} \end{cases}$$

(2) d not a square
$$h^0_{I_0,L} = \begin{cases} 0 & x \leq \frac{2\tilde{y}}{\tilde{x}+1} \\ \frac{d(\tilde{x}+1)}{2}x^2 - 2d\tilde{y} + \frac{\tilde{x}-1}{2} & \\ dx^2-1 & x \geq \frac{2\tilde{y}}{\tilde{x}-1} \end{cases}$$

$\beta(L) = \frac{\sqrt{d}}{d}$

$x \leq \frac{2\tilde{y}}{\tilde{x}+1}$

$\beta(L) \leq \frac{2y_0}{x_0+1}$ where (x_0, y_0) smallest pos sol to $x^2+4dy^2=1$

<u>Corollary</u> Improve of (N_p) for $(1,d)$