The Mathematics of Higher Symmetries

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Symmetries are very important in our modern understanding of physics.

Two cases:

Global Symmetries: Symmetry transformation is independent of coordinates

Local (gauge) symmetries: Symmetry transformation is coordinate dependent.

This Talk!

The usual (0-form) gauge symmetries are described mathematically by vector bundles:

Field $\Phi \in \Gamma(R(E))$

"Section of" Representation Bundle E

Physics	Mathematics
Field transforms non-trivially $\Updownarrow$ Field is charged	Field takes values in a non-trivial representation of E.

Abelian bundles: Charges / representations are numbers

Non-Abelian: non-trivial vector / matrix representations.

Example: $G = SU(n)$. Also referred to as the structure group of the bundle.

Field in n-dimensional fundamental: $\phi \in \Gamma(\underline{n})$

Compare / parallell transport: need connection:

$$A_\mu \in \Omega^1(Ad)$$

Adjoint representation

This is the gauge field (photon, gluon, ...)

To compare/differentiate fields: we need covariant derivative:

$$\nabla_\mu \phi = \frac{\partial \phi}{\partial x^\mu} + i A_\mu^i T_R^i \phi$$

For $\phi \in \Gamma(\underline{n})$ we have

$$\nabla_\mu \phi^a = \partial_\mu \phi^a + i A_\mu^i (T^i)^a{}_b \phi^b$$

where $(T^i)^a{}_b$ are the generators of the Lie algebra (Ad) as a matrix representation

Symmetry parameter: $\alpha \in \Gamma(\mathrm{Ad})$

(zero form)

Symmetry transformations:

$$\delta \phi_R = i\alpha^i T_R^i \phi_R$$

$$\delta A_\mu = \nabla_\mu \alpha = \partial_\mu \alpha + i[A_\mu, \alpha]$$

$$\Rightarrow \quad \delta(\nabla_\mu \phi_R) = i\alpha^i T_R^i \nabla_\mu \phi_R$$

Higher form Symmetries:

Q: What happens if α gets spacetime indecies?

Supersymmetry: α has fermionic components (spinor indecies)

What if α is a one-form, $\alpha_\mu \in \Omega^1$?

Field-Strength in Electromagnetism: $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$

This is invariant (non-Abelian: covariant) under gauge transformations:

$$\delta F_{\mu\nu} = \partial_{[\mu} \delta A_{\nu]} = \partial_{[\mu} \partial_{\nu]} \alpha = 0$$

What is the "Gauge Field" transforming under a one-form symmetry α_μ?

Answer: A two-form $B_{\mu\nu} \in \Omega^2$.

$$\delta B_{\mu\nu} = \partial_{[\mu} \alpha_{\nu]}$$

Kalb-Ramond Field!

Field-Strength: $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$

Note that $\delta H_{\mu\nu\rho} = \partial_{[\mu} \delta B_{\nu\rho]} = \partial_{[\mu} \partial_\nu \lambda_{\rho]} = 0.$

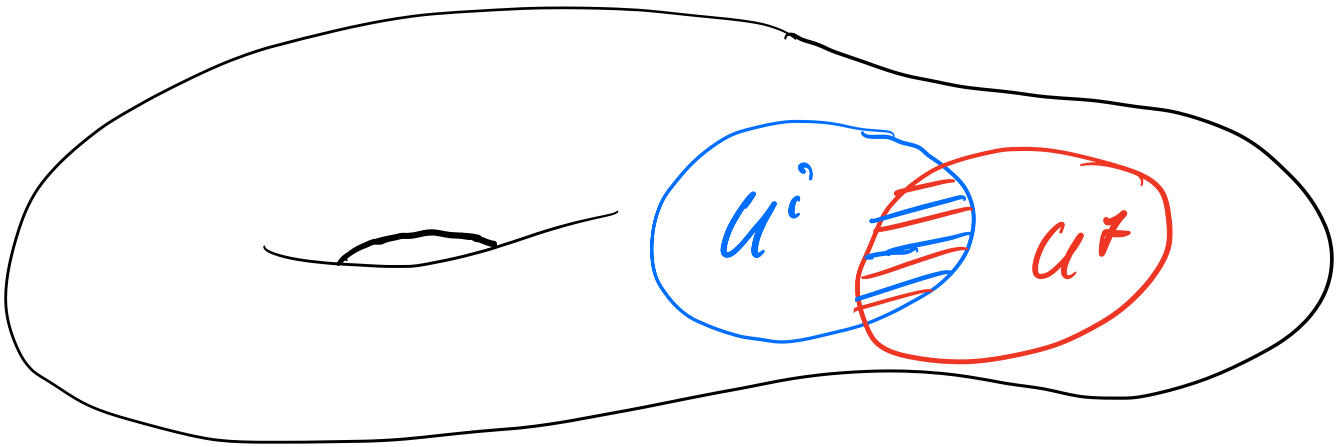
This is an Abelian one-form gauge-symmetry!

Note:
- 0-form symmetries have 0-dimensional charged objects (particles)
- 1-form symmetries have 1-dimensional charged objects (strings)
- n-form symmetries have n-dimensional charged objects (n-branes)

If 0-form symmetries are mathematically described by vector bundles, what about higher-form symmetries?

Spacetime (manifold):

M:



U^i, $U^j \approx \mathbb{R}^n$

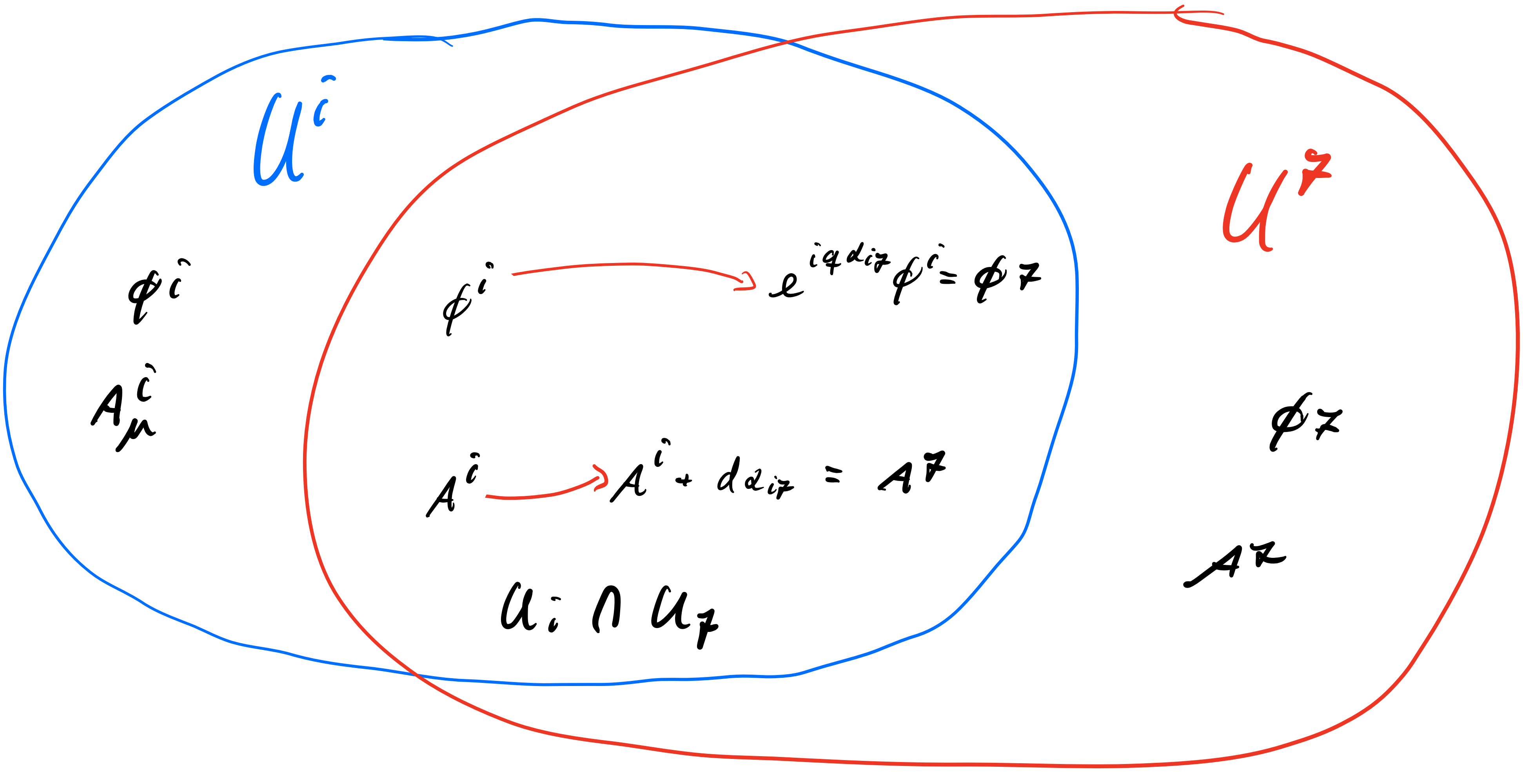
coordinate charts

$U^i \cap U^j = U^{ij}$

$\varphi_{ij} : U^{ij} \to U^{ij}$

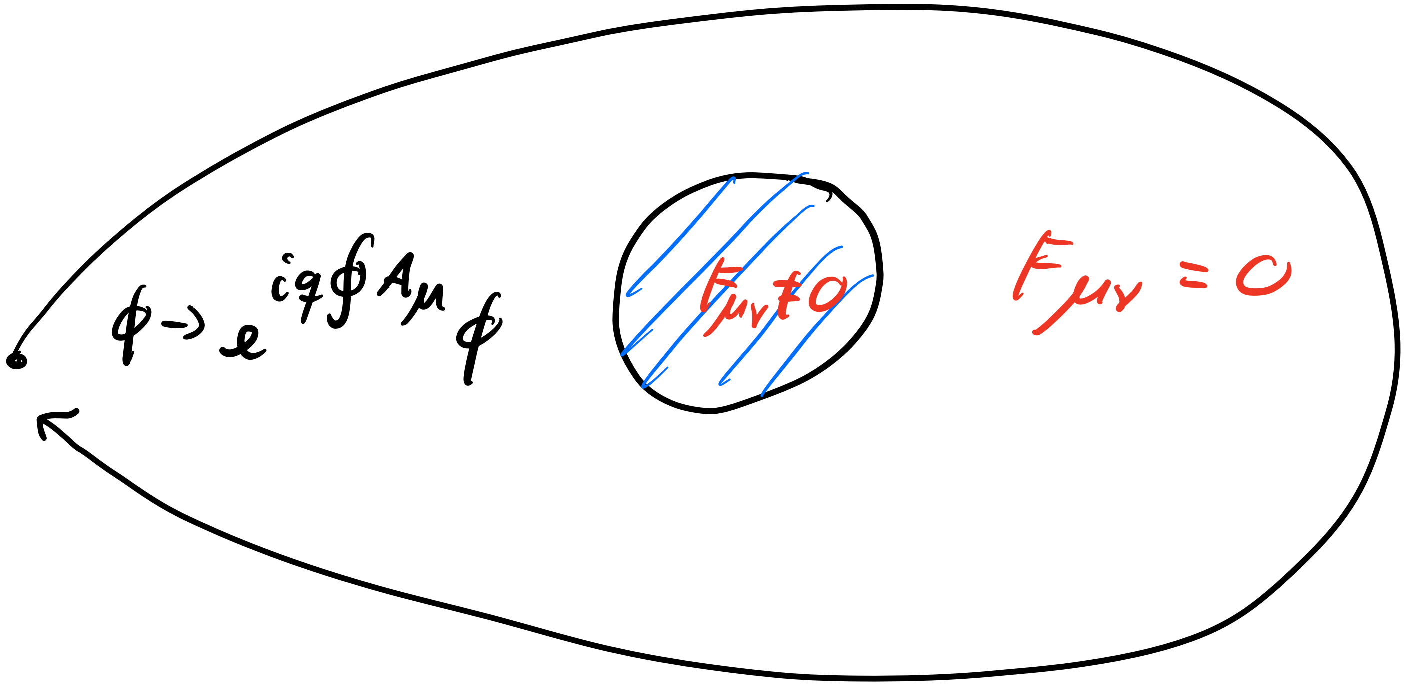
φ_{ij} : Transition functions gluing charts.

Vector bundles are patched (glued) together using local gauge transformations: $\alpha_{ij} \in \Omega^0(U^{ij})$



Ex: Aharonov - Bohm effect corresponds to a non-trivial ~~(bat~~ flat) bundle on $\mathbb{R}^2 \setminus D_2$:

$\mathbb{R}^2 \setminus D_2$:



If the bundle is trivial, then $A_\mu = \partial_\mu \alpha$ for α globally defined.

$$\Rightarrow \quad \oint A_\mu = \oint \partial_\mu \alpha = 0 .$$

If the bundle is non-trivial, then

$$A_\mu^i = \partial_\mu \alpha^i \text{ only locally on } U^i .$$

$$\text{But} \quad \oint A_\mu \overset{\text{Stokes}}{\overset{!}{=}} \int_{D_2} F_{\mu\nu}$$

Since ϕ is single-valued $\Rightarrow$ $F_{\mu\nu}$ is quantised!

Geometry: Given a bundle E over a manifold X,

$$\Rightarrow \int_{[C_i]} F \text{ is quantised}, \quad [C_i] \in H_2(X) .$$

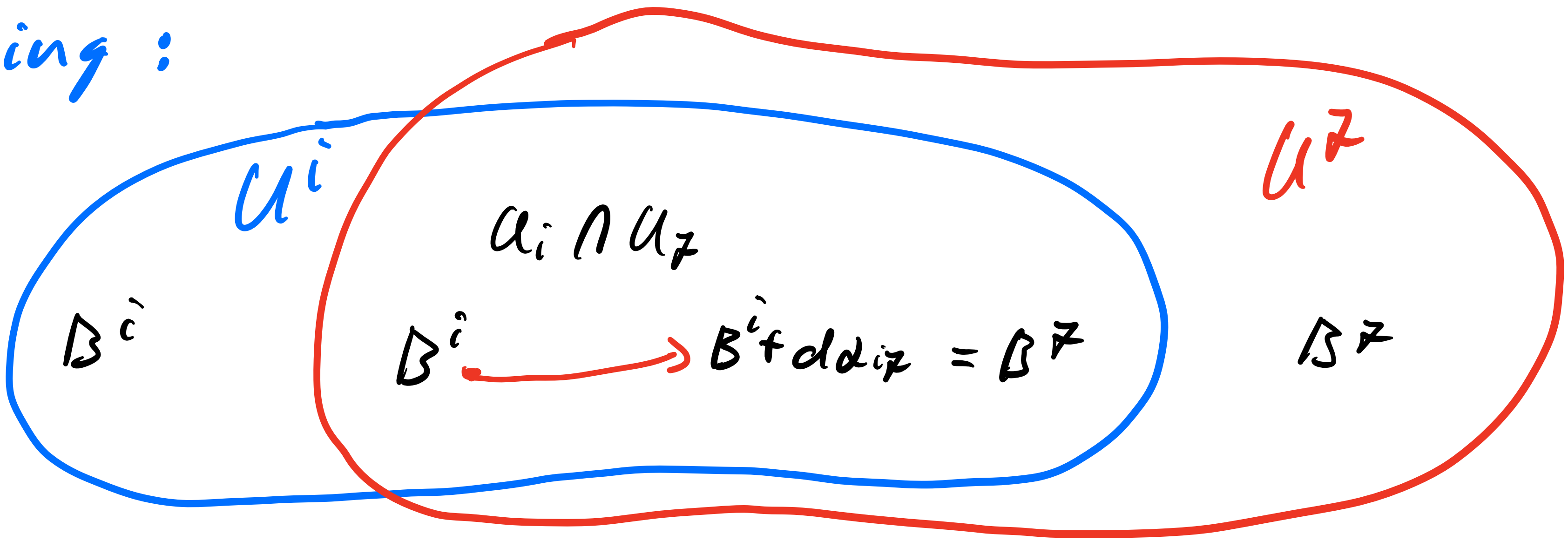
Dirac quantisation!

If the gauge-field is a two-form $B \in \Omega^2$, then the field-strength is a 3-form $H = dB \in \Omega^3$.

Quantisation: $\int_{[\gamma_i]} H$ quantised where $[\gamma_i] \in H_3(X)$.

The corresponding mathematical object is a Gerbe!

Gluing:



$\alpha_{ij} \in \Omega^1(U_i \cap U_j)$: Local gauge transf.

BUT:

Given a gauge transf. $\delta B_{\mu\nu} = \partial_{[\mu}\alpha_{\nu]}$.

$\Rightarrow$ Changing $\alpha_\mu \to \alpha_\mu + \partial_\mu \epsilon$ does not change $\delta B_{\mu\nu}$:

$$\partial_{[\mu}(\alpha_{\nu]} + \partial_{\nu]}\epsilon) = \partial_{[\mu}\alpha_{\nu]} + \underbrace{\partial_{[\mu}\partial_{\nu]}}_{\to 0}\epsilon = \delta B_{\mu\nu}.$$

$\epsilon \in \Omega^0$ is a gauge of gauge transformation!

QFT: gauge transf. $\rightsquigarrow$ ghosts
gauge of gauge $\rightsquigarrow$ ghost of ghosts!

Where do we see gauge of gauge transformations when we patch a gerbe together?

Consider Field-strength (flux) $H \in \Omega^3$

H is Global! Locally

On U^i: $H = dB^i$, $B^i \in \Omega^2(U^i)$

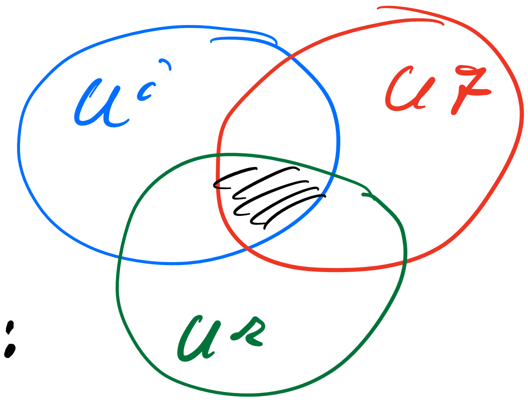
On overlap $U^i \cap U^j$: $H^i = H^j = H$

$$\Rightarrow \quad dB^i = dB^j$$

$$\Rightarrow \quad d(B^i - B^j) = 0$$

Poincaré lemma: $B^i - B^j = d\gamma^{ij}$

Here $\gamma^{ij} \in \Omega^1(U^i \cap U^j = U^{ij})$ is the local gauge transformation.



Notice: On $U^i \cap U^j \cap U^k = U^{ijk}$:

$$0 = \underbrace{B^i - B^j}_{d\gamma^{ij}} + \underbrace{B^j - B^k}_{d\gamma^{jk}} + \underbrace{B^k - B^i}_{d\gamma^{ki}} = d(\gamma^{ij} + \gamma^{jk} + \gamma^{ki})$$

$$\Rightarrow \gamma^{ij} + \gamma^{jk} + \gamma^{ki} = dc^{ijk}, \quad c^{ijk} \in \Omega^0(U^{ijk})$$

c^{ijk} are local gauge of gauge transformations

Finally: On $U^i \cap U^j \cap U^k \cap U^l = U^{ijkl}$:

$$d\left(c^{ijk} - c^{lij} + c^{kli} - c^{jkl}\right) =$$

$$dc^{ijk} - dc^{lij} + dc^{kli} - dc^{jkl} =$$

$$\gamma^{ij} + \gamma^{jk} + \gamma^{ki} - \gamma^{li} - \gamma^{ij} - \gamma^{jl}$$

$$+ \gamma^{kl} + \gamma^{li} + \gamma^{ik} - \gamma^{jk} - \gamma^{kl} - \gamma^{lj} = 0$$

$$\Rightarrow c^{ijk} + c^{lij} + c^{kli} + c^{jkl} = d^{ijkl}$$

d^{ijkl} are constant and quantised, related to flux quantisation.

Thank You !