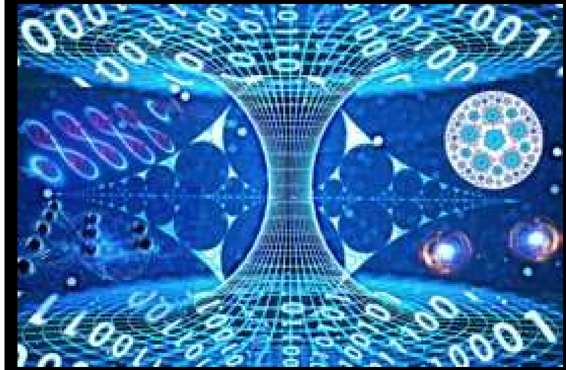


# *Quantum corrections from quotient geometry*


Victoria Martin
University of North Florida


24xx.xxxxx VM, Svesko, Zukowski, Haupfear
2312.xxxxx Bagchi, Keeler, VM, Poddar
2210.01118 VM, Poddar, Þórarinsdóttir
1811.08433 Keeler, VM, Svesko

32nd Nordic Strings   Stavanger   3 December 2023

# *Overview*

$$\det\nabla^2$$

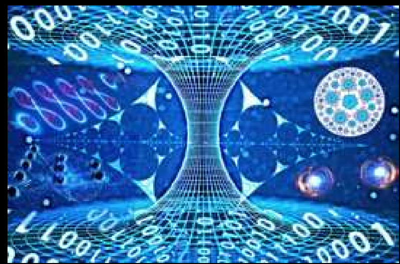
$$Z^{(1)} = \int D\phi e^{-\frac{1}{2}\int d^Dx\sqrt{g}\phi\nabla^2\phi} = \frac{\det\nabla_v^2}{\sqrt{\det\nabla_s^2\det\nabla_g^2}}$$

Classification of smooth manifolds: Can you hear the shape of a drum?

Kac 1966

$$\omega_{QN}$$

Connection between horizons and number theory?



# Overview: Calculation methods

*Heat Kernel*

$$\nabla^2 \psi_n = \lambda_n \psi_n \longrightarrow (\partial_t + \nabla_x^2) K(t; x, y) = 0$$

Giombi, Maloney, Yin 2008.

$$Z^{(1)}_{\text{scalar}} \propto \left(\det \nabla^2\right)^{-1}$$

*Quasinormal modes*

Denef, Hartnoll, Sachdev 2010.

$$Z^{(1)} = e^{\text{Pol}(\Delta)} \prod_{n,\star} \left(\omega_n - \omega_\star(\Delta)\right)^{-1}$$

*"Mystery Method"*

# Overview: Calculation methods

*Heat Kernel* $\nabla^2\psi_n = \lambda_n\psi_n \longrightarrow (\partial_t + \nabla_x^2)K(t;x,y) = 0$

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*A Selberg zeta function for quotient manifolds*

$$Z^{(1)}_{\text{reg}}(\Delta) = \frac{1}{Z_\Gamma(\Delta)} \qquad \mathcal{M}/\Gamma$$

# Not-so-mysterious method

One-loop Partition Functions of 3D Gravity

Simone Giombi[1,a], Alexander Maloney[2,b], Xi Yin[1,c]

other choices, the geometry of $H_3/\Gamma$ is more complicated. There is a rich mathematical theory – that of the Selberg trace formula and its generalizations – where the sum over elements $\gamma \in \Gamma$ is used to compute the spectrum of differential operators on $\mathbb{H}_d/\Gamma$. For scalar and vector fields, our computations precisely reproduce the results of the Selberg trace formula (see e.g. [8] and references therein). To our knowledge, the Selberg trace formula has not been successfully generalized to the graviton case.[3] Our computation may therefore be viewed as a brute force derivation of the Selberg trace formula in this context. This computation is described in section 5.

*The Selberg trace formula:*

$$\prod_n \lambda_n \to \prod_\gamma f(\gamma) \qquad \mathbb{H}_\mathrm{d}/\Gamma$$

# A Selberg zeta function for BTZ black holes

*Inspired by GMY, we discover a math paper that develops a Selberg zeta function for the BTZ black hole, calculated from the quotient structure alone.*

One-loop Partition Functions of 3D Gravity

Simone Giombi[1,a], Alexander Maloney[2,b], Xi Yin[1,c]

**SELBERG ZETA FUNCTION AND TRACE FORMULA FOR THE BTZ BLACK HOLE**

PETER A. PERRY AND FLOYD L. WILLIAMS

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - e^{2ibk_1} e^{-2ibk_2} e^{-2a(k_1+k_2+s)}\right]$$

$$a = \pi r_+$$

$$b = \pi |r_-|$$

*The payoff:*

Keeler, VM, Svesko 2018

$$Z^{(1)}_{\text{reg}}(\Delta) = \frac{1}{Z_\Gamma(\Delta)}$$

$$s^\star \leftrightarrow \omega_{QN}$$

# *A Selberg zeta function for BTZ black holes*

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**SELBERG ZETA FUNCTION AND TRACE FORMULA FOR THE BTZ BLACK HOLE**

PETER A. PERRY AND FLOYD L. WILLIAMS

**Black hole determinants and quasinormal modes**

Frederik Denef[♯,♮,♭], Sean A. Hartnoll[♯,♮] and Subir Sachdev[♯,♮]

*The payoff:*

Keeler, VM, Svesko 2018

$$Z^{(1)}_{\rm reg}(\Delta) = \frac{1}{Z_\Gamma(\Delta)}$$

$$s^\star \leftrightarrow \omega_{QN}$$

# Outline

- Heat Kernel, quotients and QNMs
- Selberg zeta functions
- BTZ black holes
- Warped AdS3 black holes
- Flat space cosmologies
- Lens Spaces
- Future Directions

# *Methods of calculating* $Z^{(1)}$

Giombi, Maloney, Yin 2008.

*Heat Kernel.* $\nabla^2 \psi_n = \lambda_n \psi_n \longrightarrow (\partial_t + \nabla_x^2) K(t; x, y) = 0$

*Scalar Field* $S^{(1)} = -\frac{1}{2} \log \det \nabla^2 = -\frac{1}{2} \sum_n \log \lambda_n = \frac{1}{2} \int_{0+}^{\infty} \frac{dt}{t} \int d^3x \sqrt{g} K(t; x, x)$

*Method of images* $K^{\mathbb{H}_3/\mathbb{Z}}(t, x, x') = \sum_{n \in \mathbb{Z}} K^{\mathbb{H}_3}\left(t, r(x, \gamma^n x')\right)$

$$-\log \det \nabla^2 = \int_0^\infty \frac{dt}{t} \int d^3x \sqrt{g} K^{\mathbb{H}_3/\mathbb{Z}}(t, x, x)$$

$$= \mathrm{vol}(\mathbb{H}_3/\mathbb{Z}) \int_0^\infty \frac{dt}{t} \frac{e^{-(m^2+1)t}}{(4\pi t)^{3/2}} + \sum_{n \neq 0} \int_0^\infty \frac{dt}{t} \int_{\mathbb{H}_3/\mathbb{Z}} d^3x \sqrt{g} K^{\mathbb{H}_3}\left(t, r(x, \gamma^n x)\right)$$

*Things to keep in mind: the infinite volume piece and the image integer n.*

# *Methods of calculating* $Z^{(1)}$

*Quasinormal modes. Two pieces of intuition:*

Denef, Hartnoll, Sachdev 2010.

1. *Relate Euclidean zero modes of the wave equation to Lorentzian QNMs via Wick rotation.*

$$\nabla^2 \phi_{\star,n} = 0$$

$$\omega_{QN}\left(\Delta_{\star,n}\right) = \omega_n$$

2. *Weierstrass factorization theorem: A meromorphic function is characterized by its zeros and poles.*

$$Z^{(1)} = e^{\mathrm{Pol}(\Delta)} \prod_{n,\star} \left(\omega_n - \omega_\star(\Delta)\right)^{-1}$$

# *Methods of calculating* $Z^{(1)}$

*Heat Kernel*

Giombi, Maloney, Yin 2008.

$$-\log\det\nabla^2 = \int_0^\infty \frac{dt}{t}\int d^3x\sqrt{g}K^{\mathbb{H}_3/\mathbb{Z}}(t,x,x)$$

$$= \mathrm{vol}(\mathbb{H}_3/\mathbb{Z})\int_0^\infty \frac{dt}{t}\frac{e^{-(m^2+1)t}}{(4\pi t)^{3/2}} + \sum_{n\neq 0}\int_0^\infty \frac{dt}{t}\int_{\mathbb{H}_3/\mathbb{Z}} d^3x\sqrt{g}K^{\mathbb{H}_3}\left(t, r(x,\gamma^n x)\right)$$

*Quasinormal modes*

Denef, Hartnoll, Sachdev 2010.

$$\omega_{QN}\left(\Delta_{\star,n}\right) = \omega_n \qquad Z^{(1)} = e^{\mathrm{Pol}(\Delta)}\prod_{n,\star}\left(\omega_n - \omega_\star(\Delta)\right)^{-1}$$

*Zeta function from quotient group. Can calculate 1-loop partition function without heat kernel or QNMs.*

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left[1 - e^{2ibk_1}e^{-2ibk_2}e^{-2a(k_1+k_2+s)}\right]$$

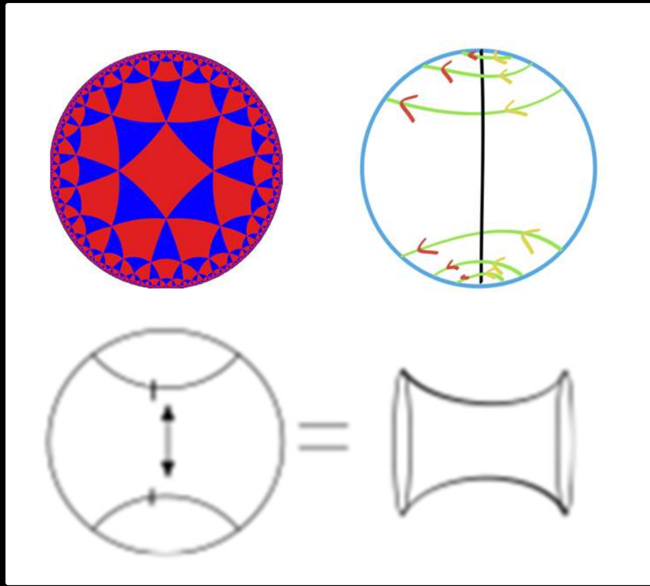
$$Z_\Gamma(s^\star) = 0 \qquad \exp = 2\pi i n \qquad s^\star \leftrightarrow \omega_{QN}$$

# *Main Idea*

A quotient geometry (or orbifold) is made by identifying points in empty spacetime.

$$\mathcal{M}/\Gamma \qquad \Gamma \in \mathrm{SL}(2,\mathbb{R})$$

MANY interesting examples: BTZ black hole, warped AdS black holes, k-boundary wormholes, flat space cosmologies... $\Gamma \sim \mathbb{Z}$



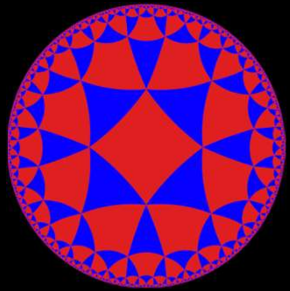
What can you learn from the quotient structure alone? $\omega_{QN}$ $Z^{(1)}_{reg}$

Mechanism: we construct a Selberg-like zeta function from the quotient generators. This is a cousin of the Riemann zeta function $s^\star \leftrightarrow \omega_{QN}$

# *Motivation from Mathematics*

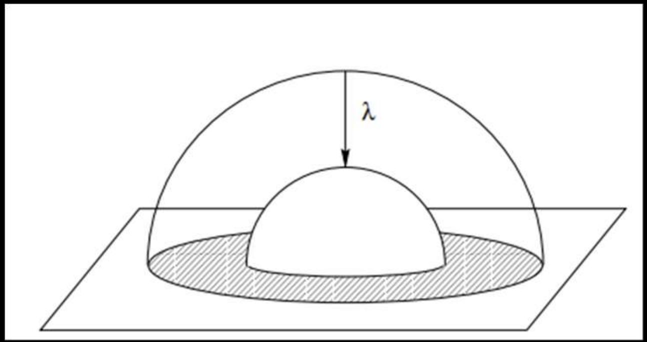
From a mathematician's point of view, the Selberg zeta function and trace formula are only defined for hyperbolic quotients.

$\mathbb{H}^n/\Gamma$



The Selberg trace formula relates the spectrum of a differential operator on a hyperbolic quotient manifold to geometric (group theoretic) data on that manifold.

$$\det \nabla^2 \sim \prod_{\gamma \in \Gamma_{\mathrm{pcc},\star}} f_\star(\gamma)$$




Krasnov 2000

The pcc product is equivalent to a product over primitive geodesics on the spacetime

Excellent reference for physicists: Gutzwiller, *Chaos in Classical and Quantum Mechanics*, 1990.

Our question: How far can we extend this formalism? $\mathcal{M}/\Gamma$

# Fun with zeta functions

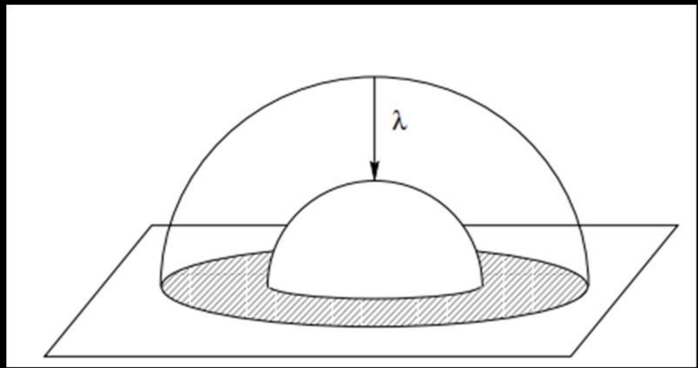
Riemann zeta function: $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

Selberg zeta function (hyperbolic quotients): Replace prime numbers with prime geodesics.

$$Z_\Gamma(s) = \prod_{p} \prod_{n=0}^{\infty} \left(1 - N(p)^{-s-n}\right)$$

meromorphic function on the complex plane

Prime geodesics: closed geodesics that trace out their path exactly once. They are conjugacy classes of primitive hyperbolic elements of the quotienting group.

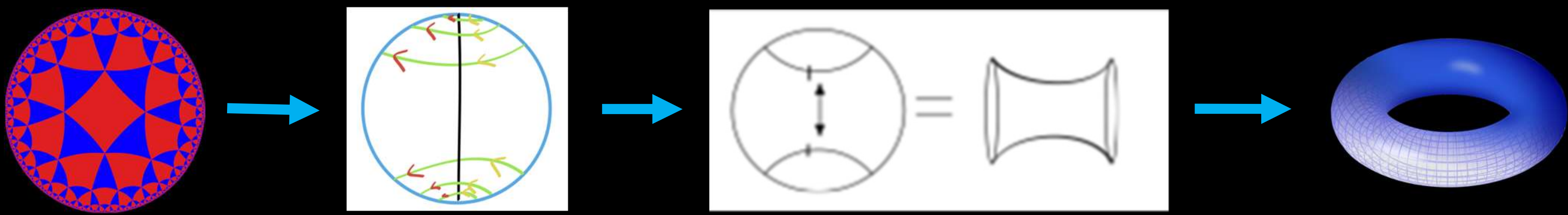


Krasnov 2000

N(p) is a function of the length of the prime geodesic. It is the norm of the primitive conjugacy class of hyperbolic isometries. For us it will be: $e^{\ell}$

# *Quotient structure of BTZ*

## *BTZ can be constructed by folding up empty AdS3*



$$ds^2 = \frac{L^2}{z^2}\left(dx^2 + dy^2 + dz^2\right) \longrightarrow ds^2 = N(r)^2 d\tau^2 + \frac{dr^2}{N(r)^2} + r^2\left(N^\phi(r)d\tau + d\phi\right)^2$$

### *Coordinate transformation*

$$x = A(r)\cos f(\phi,\tau)\exp(r_+\phi - |r_-|\tau)$$
$$y = A(r)\sin f(\phi,\tau)\exp(r_+\phi - |r_-|\tau)$$
$$z = B(r)\exp(r_+\phi - |r_-|\tau)$$

### *Quotient Group*

$$\phi \rightarrow \phi + 2\pi n$$
$$\gamma^n \cdot (x,y,z) = (x',y',z')$$

*Group element*

See, for example: Banados, Henneaux, Teitelboim, Zanelli 1993; Carlip 1995.

# Quotient structure of BTZ

## Coordinate transformation

$$x = A(r)\cos f(\phi,\tau)\exp(r_+\phi - |r_-|\tau)$$
$$y = A(r)\sin f(\phi,\tau)\exp(r_+\phi - |r_-|\tau)$$
$$z = B(r)\exp(r_+\phi - |r_-|\tau)$$

## Quotient Group

$$\phi \to \phi + 2\pi n$$
$$\gamma^n \cdot (x,y,z) = (x',y',z')$$

## Group action: Dilation and Rotation

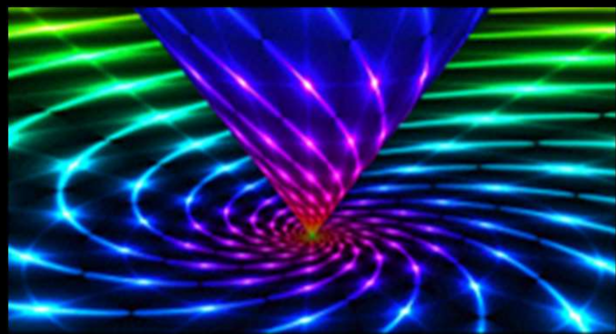
$$\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cos 2b & -\sin 2b & 0 \\ \sin 2b & \cos 2b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$a = \pi r_+$$
$$b = \pi |r_-|$$

Group generator

$$\pi\partial_\phi = aJ_{12} + bJ_{03}$$

$$J_{12} = x\partial_x + y\partial_y + z\partial_z \qquad J_{03} = x\partial_y - y\partial_x$$



# BTZ Selberg zeta function

Perry, Williams 2003

**BTZ:** $$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left[1 - e^{-2\pi i(\tau_1(k_1-k_2)+i\tau_2(k_1+k_2+s))}\right] \qquad \tau = \tau_1 + i\tau_2$$

$$a = \pi\tau_2 = \pi r_+/L \qquad b = -\pi\tau_1 = -\pi r_-/L$$

$$\pi\partial_\phi = aJ_{12} + bJ_{03}$$

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left[1 - q^{k_2+s/2}\bar{q}^{k_1+s/2}\right] \qquad q = e^{2\pi i\tau}$$

BTZ Selberg zeta has zeros s* when the exponent is $2\pi i\ell$

$$s^\star = \Delta \quad \leftrightarrow \quad \omega_{QN} = \omega_n \qquad \log Z_\Gamma(\Delta) \propto \log\det\nabla^2_{reg}$$

Keeler, VM, Svesko 2018

# *BTZ Selberg zeta function*

*Perry, Williams 2003*

*BTZ:*

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - e^{-2\pi i(\tau_1(k_1-k_2)+\tau_2(k_1+k_2+s))}\right]$$

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - q^{k_2+s/2}\bar{q}^{k_1+s/2}\right]$$

$$\tau = \tau_1 + i\tau_2$$

$$q = e^{2\pi i\tau}$$

$$a = \pi\tau_2 = \pi r_+/L$$

$$b = -\pi\tau_1 = -\pi r_-/L$$

*BTZ Selberg zeta has zeros s* when the exponent is* $2\pi i\ell$

$$s^\star = \Delta \quad \leftrightarrow \quad \omega_{QN} = \omega_n \qquad \log Z_\Gamma(\Delta) \propto \log\det\nabla^2_{reg}$$

*Keeler, VM, Svesko 2018*

*Integers correspond to QNM quantum numbers*

$(k_1 + k_2) \sim$ radial quantum number

$(k_1 - k_2) \sim$ thermal quantum number

$\ell \sim$ angular quantum number

# A word about generators

To extract the Selberg zeta function parameters, we use the embedding generators

$$\pi\partial_\phi = aJ_{12} + bJ_{03}$$

$$ds^2 = -dU^2 + dV^2 + dX^2 + dY^2 \qquad -U^2 + V^2 + X^2 + Y^2 = -L^2$$

Poincare patch embedding

$$x = \frac{Y}{U+X}, \qquad y = \frac{V}{U+X}, \qquad z = \frac{L}{U+X}$$

6 isometry generators

$$J_{AB} = X_B\partial_A - X_A\partial_B \qquad SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

$$J_{12} = x\partial_x + y\partial_y + z\partial_z \longrightarrow \mathcal{L}_0 + \bar{\mathcal{L}}_0$$

$$J_{03} = x\partial_y - y\partial_x \longrightarrow \mathcal{L}_0 - \bar{\mathcal{L}}_0$$

# Warped Black Holes: $WAdS_3/\Gamma$

WAdS3 $$ds^2 = \frac{L^2}{\nu^2+3}\left(-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du+\sinh\sigma d\tau)^2\right)$$

Warped AdS black holes are quotients of Warped AdS, with the same identification, but different parameters.

Anninos, Li, Padi, Song, Strominger 2008

$$\partial_\phi = \frac{\nu^2+3}{8}\left[\left(r_+ - r_- - \frac{\sqrt{(\nu^2+3)r_+r_-}}{\nu}\right)J_2 - (r_+ - r_-)\tilde{J}_2\right]$$

$$J_2 \sim \mathcal{L}_0$$
$$\tilde{J}_2 \sim \bar{\mathcal{L}}_0$$

The Selberg-like zeta function takes the same form as that for BTZ:

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left[1 - e^{-2\pi i(\tau_1(k_1-k_2)+\tau_2(k_1+k_2+s))}\right]$$

VM, Poddar, Thorarinsdottir 2022

$$\partial_\phi = aJ_{12} + bJ_{03} \qquad a = \frac{\pi r_+(\nu^2+3)(2\nu r_+ - r_-\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)} \qquad b = a(r_+ \leftrightarrow r_-)$$

# Strategy: "conformal" coordinates

If we wish to proceed in the same manner as Perry and Williams, we run into an issue. We want an analogous coordinate transformation:

$$(\tau, r, \phi) \in \mathbb{R}^3 \quad \leftrightarrow \quad (x, y, z) \in W\mathbb{H}^3$$

The issue: we don't have a warped version of the hyperbolic half plane as a target metric .

$$ds^2 = \frac{L^2}{z^2}\left(dx^2 + dy^2 + dz^2\right)$$

Strategy: Propose a coordinate transformation ansatz. Our task is to find the appropriate $(\alpha, \beta, \gamma, \delta)$ that capture the symmetries of our warped quotient.

$$w^+ = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}\, e^{\alpha\phi + \beta t}$$

$$w^- = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}\, e^{\gamma\phi + \delta t}$$

$$w^+ = x + iy \qquad w^- = x - iy$$

$$z = \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}}\, e^{1/2((\alpha+\gamma)\phi + (\beta+\delta)t)}$$

# Strategy: "conformal" coordinates

From these coordinates we can make six generators:

$$H_1 = i\partial_+, \qquad H_0 = i\left(w^+\partial_+ + \frac{1}{2}z\partial_z\right), \qquad H_{-1} = i((w^+)^2\partial_+ + w^+z\partial_z - z^2\partial_-)$$

$$\bar{H}_1 = i\partial_-, \qquad \bar{H}_0 = i\left(w^-\partial_- + \frac{1}{2}z\partial_z\right), \qquad \bar{H}_{-1} = i((w^-)^2\partial_- + w^-z\partial_z - z^2\partial_+)$$

They form two copies of the SL(2,R) algebra: $[H_i, H_j] = (i-j)H_{i+j}$

$$i, j \in \{0, \pm 1\}$$

Each set has the quadratic Casimir:

$$\begin{aligned}\mathcal{H}^2 &= -\bar{H}_0^2 + \frac{1}{2}(\bar{H}_1\bar{H}_{-1} + \bar{H}_{-1}\bar{H}_1) \\ &= \frac{1}{4}(z^2\partial_z^2 - z\partial_z) + z^2\partial_+\partial_-\end{aligned}$$

Our warped quotient has symmetry group SL(2,R)xU(1) . This symmetry reflected in the wave equation as well!

$$\nabla^2\Phi = 0 \qquad \Phi = R(r)e^{i(k\phi - \omega t)}$$

# Compare Laplacian and Casimir

Now, to find our appropriate $(\alpha, \beta, \gamma, \delta)$ , we exploit the symmetry of our wave equation and conclude that the SL(2,R)xU(1) Casimir is proportional to the scalar Laplacian:

$$(\mathcal{H}^2 + \lambda H_0^2)\Phi \propto \nabla^2 \Phi$$

Now compare!

$$(\mathcal{H}^2 + \lambda H_0^2)R(x)$$

$$= \left( \partial_x \left( x^2 - \frac{1}{4} \right) \partial_x + \frac{(\omega(\alpha+\gamma) + k(\beta+\delta))^2}{4\left(x - \frac{1}{2}\right)(\beta\gamma - \alpha\delta)^2} - \frac{(\omega(\alpha-\gamma) + k(\beta-\delta))^2}{4\left(x + \frac{1}{2}\right)(\beta\gamma - \alpha\delta)^2} + \lambda \frac{(k\delta + \gamma\omega)^2}{(\beta\gamma - \alpha\delta)^2} \right) R(x)$$

versus

$$\nabla^2 = \partial_x \left( x^2 - \frac{1}{4} \right) \partial_x + \frac{P}{4\left(x - \frac{1}{2}\right)} + \frac{Q}{4\left(x + \frac{1}{2}\right)} + S$$

$$P = \frac{4\left(k\left(-\left(\sqrt{\nu^2+3} - 4\right) r_+ r_- + 2(\nu-1) r_+^2 - 2r_-^2\right) - \omega r_+ \left(2\nu r_+ - \sqrt{\nu^2+3}\, r_-\right)\right)^2}{(r_+ - r_-)^2 (r_+ + r_-)^2 (3+\nu^2)^2}$$

$$x = \frac{r^2 - 1/2(r_+^2 + r_-^2)}{r_+^2 - r_-^2}$$

# Some results

$$\alpha = \frac{\left(\nu^2+3\right)\left(\nu(r_+^2+r_-^2) - r_+ r_- \sqrt{\nu^2+3}\right)}{4\nu(r_+ - r_-)} \qquad \gamma = \frac{1}{4}\left(\nu^2+3\right)\left(r_- + r_+\right)$$

$$\beta = \frac{(r_+ - r_-)(3+\nu^2)}{2} - \frac{\left(\nu^2+3\right)\left(\nu(r_+^2+r_-^2) - r_+ r_- \sqrt{\nu^2+3}\right)}{4\nu(r_+ - r_-)} \qquad \delta = -\frac{1}{4}\left(\nu^2+3\right)\left(r_- + r_+\right)$$

Now, let's see what metric our coordinate transformation makes!

$$w^+ = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}\, e^{\alpha\phi + \beta t} \qquad w^- = \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}\, e^{\gamma\phi + \delta t} \qquad z = \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}}\, e^{1/2((\alpha+\gamma)\phi + (\beta+\delta)t)}$$

$$ds^2 = \frac{4}{(3+\nu)^2 z^2}\left((3+\nu^2)dw_+ dw_- + 4\nu^2 dz^2 + \frac{3(\nu^2-1)w_+}{z^2}(dw_-^2 + 2z\,dw_- dz)\right)$$

This metric has four isometries, generated by:

$$\bar{H}_0 = i(w_-\partial_- + \frac{1}{2}z\partial_z) \qquad \bar{H}_{-1} = i(-z^2\partial_+ + {w^-}^2\partial_- + w^- z\partial_z)$$

$$H_0 = i(w_+\partial_+ + \frac{1}{2}z\partial_z) \qquad \bar{H}_1 = i\partial_-$$

# Some results

This metric has four isometries, generated by:

$$\bar{H}_0 = i(w_-\partial_- + \frac{1}{2}z\partial_z) \qquad \bar{H}_{-1} = i(-z^2\partial_+ + w^{-2}\partial_- + w^- z\partial_z)$$

$$H_0 = i(w_+\partial_+ + \frac{1}{2}z\partial_z) \qquad \bar{H}_1 = i\partial_-$$

The generators corresponding to rotation in the $w^{\pm}$ plane and dilation are:

$$H_0 - \bar{H}_0 = i(w_+\partial_+ - w_-\partial_-)$$

$$H_0 + \bar{H}_0 = i(w_+\partial_+ + w_-\partial_- + z\partial_z)$$

The quotient is generated by the group element:

$$e^{-2\pi i(\gamma H_0 + \alpha \bar{H}_0)} = e^{2\pi\partial_\phi}$$

To get a and b for our Selberg zeta function, we need to find out the coefficients of dilation and rotation:

$$e^{-2\pi i\left(\frac{\gamma+\alpha}{2}(H_0+\bar{H}_0)+\frac{\gamma-\alpha}{2}(H_0-\bar{H}_0)\right)} \qquad 2a = \gamma + \alpha \qquad 2b = \gamma - \alpha$$

# Warped Black Holes: $WAdS_3/\Gamma$

The Selberg-like zeta function takes the same form as that for BTZ:

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left[1 - e^{-2\pi i(\tau_1(k_1-k_2)+\tau_2(k_1+k_2+s))}\right]$$

VM, Poddar, Thorarinsdottir 2022

$$\partial_\phi = aJ_{12} + bJ_{03} \quad a = \frac{\pi r_+(\nu^2+3)(2\nu r_+ - r_-\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)} \quad b = a(r_+ \leftrightarrow r_-)$$

Reproduce QNMs for Warped AdS Black holes!

Chen, Xu 2009, Ferreria 2013

Candidate 1-loop scalar partition function for warped AdS3 black holes $Z^{(1)}_{\rm scalar}$

Proof of concept for Selberg zeta techniques beyond hyperbolic quotients

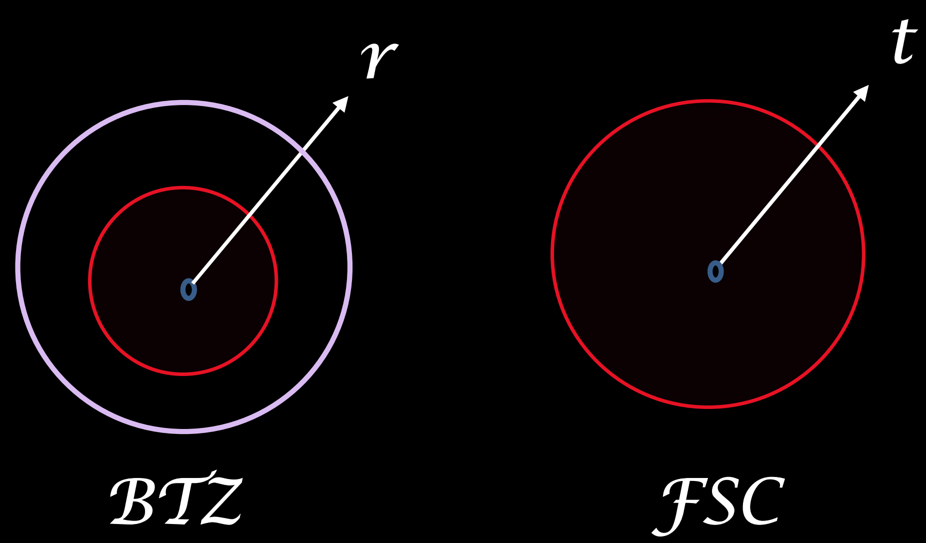
# Flat Space Cosmologies

$$ds^2_{\text{BTZ}} = \left(8GM - \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{-8GM + \frac{r^2}{L^2} + \frac{16G^2J^2}{r^2}} - 8GJdtd\phi + r^2d\phi^2$$

FSC as a limit of BTZ: $r_+ \to L\sqrt{8GM} = L\hat{r}_+ \quad r_- \to \sqrt{\frac{2G}{M}}|J| = r_0 \quad G/L \to 0$

$$ds^2_{\text{FBTZ}} = \hat{r}_+^2 dt^2 - \frac{r^2\,dr^2}{\hat{r}_+^2(r^2 - r_0^2)} + r^2d\phi^2 - 2\hat{r}_+ r_0 dtd\phi$$

The radial coordinate is now timelike, and we have a timelike (cosmological) horizon.




Let's try and calculate QNMs using Selberg techniques!

# Flat Space Cosmologies

AdS3: Asymptotic symmetry algebra (ASA) formed by two commuting copies of Virasoro:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

3D Minkowski: ASA at null infinity is BMS3

$$[L_m, L_n] = (m-n)L_{m+n} + c_{LL}m(m^2-1)\delta_{m+n,0}$$

$$[L_m, M_n] = (m-n)M_{m+n} + c_{LM}m(m^2-1)\delta_{m+n,0}$$

In the limit from AdS3 to flat space:

$$L_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \qquad M_n = \epsilon\left(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}\right), \qquad \epsilon = G/L \to 0$$

Representations of BMS3:

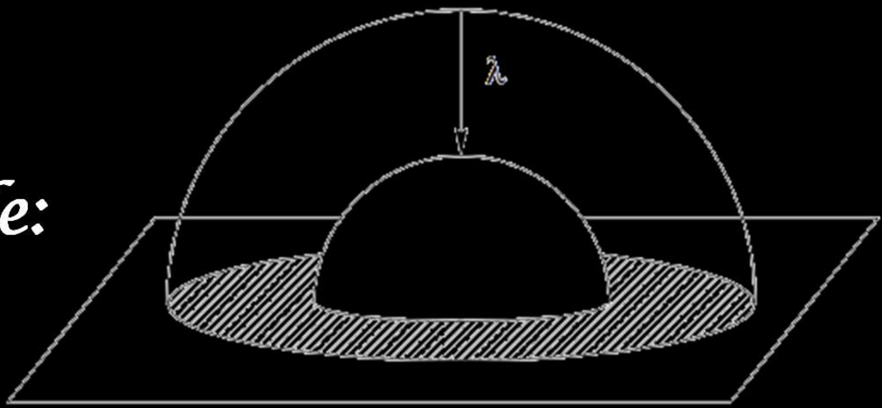
$$L_0 \left|m, j\right\rangle = j\left|m, j\right\rangle, \qquad M_0\left|m, j\right\rangle = m\left|m, j\right\rangle$$

$$j = h - \bar{h}, \qquad m = \lim_{\epsilon\to 0}\epsilon(h+\bar{h})$$

Bagchi, Detourney, Fareghbal, Simón 2012

# FSC: a quotient of flat space

Recall BTZ example:




$$\pi\partial_\phi = aJ_{12} + bJ_{03}$$

$$J_{12} = x\partial_x + y\partial_y + z\partial_z$$

$$J_{03} = x\partial_y - y\partial_x$$

$$a = \pi\tau_2 = \pi r_+/L$$

$$b = -\pi\tau_1 = -\pi r_-/L$$

Flat limit: $r_+ \to L\hat{r}_+ \qquad r_- \to r_0 \qquad G/L \to 0$

$$L_0 = \lim_{L\to\infty} J_{12} = X\partial_T + T\partial_X$$

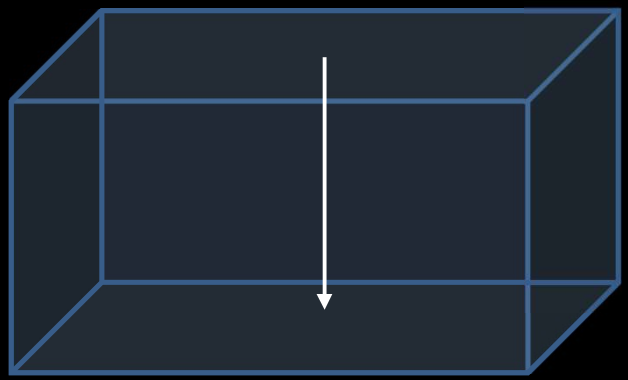
$$\frac{M_0}{G} = \lim_{L\to\infty} \frac{1}{L} J_{03} = \partial_Y$$

Cornalba, Costa 2002

FSCs are "shifted boost orbifolds"

# FSC: a quotient of flat space

Flat limit: $r_+ \to L\hat{r}_+ \qquad r_- \to r_0 \qquad G/L \to 0$



$$L_0 = \lim_{L\to\infty} J_{12} = X\partial_T + T\partial_X$$

$$\frac{M_0}{G} = \lim_{L\to\infty} \frac{1}{L} J_{03} = \partial_Y$$

Cornalba, Costa 2002

FSC generator from BTZ:

$$\pi\partial_\phi = \boxed{aJ_{12}} + \boxed{bJ_{03}}$$

$$a = \pi\tau_2 = \pi r_+/L$$

$$b = -\pi\tau_1 = -\pi r_-/L$$

$aJ_{12} \to \pi\hat{r}_+ L_0$

$bJ_{03} \to Lb\,\frac{J_{03}}{L} \to -\pi r_0 \frac{M_0}{G}$

$$\boxed{\partial_\phi = \eta L_0 + \rho M_0}$$

$$\eta = \hat{r}_+$$

$$\rho = -\frac{r_0}{G}$$

# *Generalized Selberg zeta.* $\mathcal{M}_3/\mathbb{Z}$

$$Z_\Gamma(s) = \prod_{\text{descendants}} \left\langle 1 - e^{2\pi i \partial_\phi} \right\rangle_{\text{scalar primary of weight } s}$$

*BTZ* *Primary:* $|h, \bar{h}\rangle$ *Descendants:* $(\mathcal{L}_{-1})^{k_2} |h, \bar{h}\rangle = |h, \bar{h}, k_2\rangle$

$$(\bar{\mathcal{L}}_{-1})^{k_1} |h, \bar{h}\rangle = |h, \bar{h}, k_1\rangle$$

*Group element*

$$e^{2\pi i \partial_\phi} = e^{2\pi i((\mathcal{L}_0 - \bar{\mathcal{L}}_0)\tau_1 + (\mathcal{L}_0 + \bar{\mathcal{L}}_0) i\tau_2)} = q^{\mathcal{L}_0} \bar{q}^{\bar{\mathcal{L}}_0}$$

*Eigenvalues*

$$\mathcal{L}_0 |h, \bar{h}, k_1, k_2\rangle = (h + k_2) |h, \bar{h}, k_1, k_2\rangle = \left(\frac{\Delta}{2} + k_2\right) |h, \bar{h}, k_1, k_2\rangle$$

$$\bar{\mathcal{L}}_0 |h, \bar{h}, k_1, k_2\rangle = (\bar{h} + k_1) |h, \bar{h}, k_1, k_2\rangle = \left(\frac{\Delta}{2} + k_1\right) |h, \bar{h}, k_1, k_2\rangle$$

*Remember: We would like to identify* $s \leftrightarrow \Delta$

# Generalized Selberg zeta. $\mathcal{M}_3/\mathbb{Z}$

$$Z_\Gamma(s) = \prod_{\text{descendants}} \left\langle 1 - e^{2\pi i \partial_\phi} \right\rangle_{\text{scalar primary of weight } s}$$

Group element

$$e^{2\pi i \partial_\phi} = e^{2\pi i ((\mathcal{L}_0 - \bar{\mathcal{L}}_0)\tau_1 + (\mathcal{L}_0 + \bar{\mathcal{L}}_0) i \tau_2)} = q^{\mathcal{L}_0} \bar{q}^{\bar{\mathcal{L}}_0}$$

Putting this all together, we recover the BTZ Selberg zeta function.

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2\pi i ((k_2 - k_1)\tau_1 + (k_1 + k_2 + s) i \tau_2)}\right)$$

Based on this and previous work, we can conjecture a schematic Selberg zeta function for more general settings.

$\mathcal{M}/\Gamma$

$$Z_\Gamma(s) = \prod_{\gamma} \prod_{k_1,k_2,\ldots} \langle 1 - \gamma \rangle$$

# *Selberg zeta for FSC*

$$Z_\Gamma(s) = \prod_{\text{descendants}} \left\langle 1 - e^{2\pi i \partial_\phi} \right\rangle_{\text{scalar primary of weight } s}$$

*In terms of the BMS generators, the group element is:*

$$e^{2\pi i \partial_\phi} = e^{2\pi i (L_0 \eta + M_0 \rho)}$$

*A primary field of mass m and its descendants:*

$$M_0 \, |m, k_1 - k_2\rangle = m \, |m, k_1 - k_2\rangle$$

$$L_0 \, |m, k_1 - k_2\rangle = (k_1 - k_2) \, |m, k_1 - k_2\rangle$$

*FSC Selberg-like zeta function:*

$$Z_\Gamma(s) = \prod_{k_1, k_2 = 0}^{\infty} \left( 1 - e^{2\pi i (\eta (k_1 - k_2) + s\rho)} \right)$$

*Remember: We would like to identify* $s \leftrightarrow m$

# *Selberg zeta for FSC*

$$Z_\Gamma(s) = \prod_{\text{descendants}} \langle 1 - e^{2\pi i \partial_\phi} \rangle_{\text{scalar primary of weight } s}$$

*FSC Selberg-like zeta function:*

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2\pi i(\eta(k_1-k_2)+s\rho)}\right)$$

*This answer makes sense from a QNM standpoint!*

~~$(k_1 + k_2) \sim$ radial quantum number~~

$(k_1 - k_2) \sim$ thermal quantum number

$\ell \sim$ angular quantum number

*If we expect the Selberg zeros to produce the QNMs, then we should not expect them to contain a radial quantum number (since our horizon is timelike).*

# Check 1: Partition function

Happily, the scalar one-loop partition function for FSC has already been computed by other means!

Barnich, Gonzalez, Maloney, Oblak 2015

$$Z^{\text{1-loop}}_{\text{flat, scalar}}(m) = (\det \nabla^2_{\text{flat, scalar}})^{-\frac{1}{2}} = \exp\left(\sum_{n=1}^{\infty} \frac{e^{2\pi i m n \rho}}{n|(1-e^{2\pi i n \eta})|^2}\right)$$

From this we can recover the Selberg zeta function in the usual way.

$$\xi = e^{2\pi i \rho} \qquad \chi = e^{2\pi i \eta}$$

$$Z^{\text{1-loop}}_{\text{flat, scalar}}(m) = \exp\left(\sum_{n=1}^{\infty} \frac{\xi^{mn}}{n|(1-\chi^n)|^2}\right) = \exp\left(\sum_{n=1}^{\infty} \sum_{k_1,k_2=0}^{\infty} \frac{1}{n}(\xi^m \chi^{k_1} \bar{\chi}^{k_2})^n\right)$$

$$= \prod_{k_1,k_2=0}^{\infty} \frac{1}{1-\xi^m \chi^{k_1} \bar{\chi}^{k_2}}$$

We recover $Z_\Gamma(s) = (\det \nabla^2_{\text{flat, scalar}})^{\frac{1}{2}} = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2\pi i (s\rho + (k_1-k_2)\eta)}\right)$ ✓

# *Check 2: Limit from BTZ*

*Start with the BTZ Selberg zeta function*

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left(1 - e^{2\pi i((k_2-k_1)\tau_1+(k_1+k_2+s)i\tau_2)}\right)$$

*To take the limit, employ a change of basis*

$$\eta = \frac{\tau+\bar{\tau}}{2} \qquad \rho = \left(\frac{\tau-\bar{\tau}}{2}\right)$$

***Recall:*** $L_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \qquad M_n = \epsilon\left(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}\right), \qquad \epsilon = G/L \to 0$

$$\partial_\phi = \eta L_0 + \rho M_0 \qquad M_0\,|m, k_1 - k_2\rangle = m\,|m, k_1-k_2\rangle \qquad s \leftrightarrow m$$

*Thus, we should employ the scaling:* $s \to \frac{s}{\epsilon}, \;\; \rho \to \epsilon\rho$

*The limit gives our zeta function*

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty}\left(1 - e^{2\pi i\left(\eta(k_1-k_2)+\epsilon\rho(k_1+k_2+\frac{s}{\epsilon})\right)}\right)$$

$$= \prod_{k_1,k_2=0}^{\infty}\left(1 - e^{2\pi i(\eta(k_1-k_2)+s\rho)}\right)$$

✓

# Quick slide to recap what we want and need: QNMs and thermal frequencies.

$$s^\star = \Delta \quad \leftrightarrow \quad \omega_{QN} = \omega_n$$

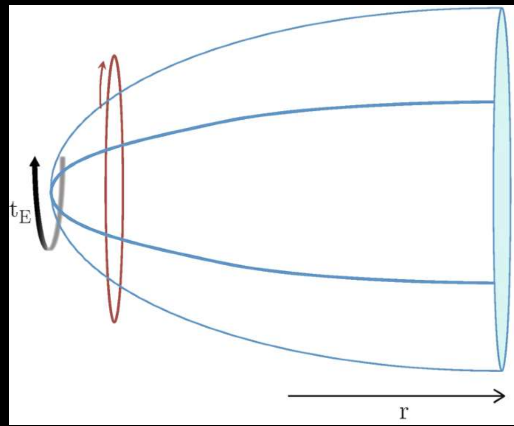
For FSC, conformal dimension is m

We need to calculate the thermal frequencies

# FSC Thermal frequencies

*In order to find the FSC QNMs from the zeta function, we need to find compute the thermal frequencies.*

*Thermal frequencies are computed by insisting that our probe fields are regular at the horizon.*




$$ds^2 = -\frac{1}{\hat{r}_+}\frac{r^2}{r^2 - r_0^2}dr^2 + \frac{\hat{r}_+^2(r^2 - r_0^2)}{r^2}dv^2 + r^2\left(d\phi - \frac{\hat{r}_+ r_0}{r^2}dv\right)^2$$

$$\Phi(r, v, \phi) = e^{i(k\phi - \omega v)}f(r) \quad (r - r_0)^2 f''(r) + \frac{r^4 - r_0^4}{r(r + r_0)^2}f'(r) + \frac{r^2(k^2\hat{r}_+^2 - 2\hat{r}_+ r_0 k\omega + r^2\omega^2)}{\hat{r}_+^4(r + r_0)^2}f(r) = 0$$

$$\boxed{\omega_n = \Omega k + 2\pi i n T} \quad \Omega = \frac{\hat{r}_+}{r_0} \quad T = \frac{\hat{r}_+^2}{2\pi r_0}$$

*Thermal frequencies can also be obtained as a limit of BTZ values.*

# FSC QNMs

*Mechanism for determining QNMs:* $s^* \leftrightarrow \Delta$ $\omega_{QN} \leftrightarrow \omega_n$

*We can write this condition as:*

$$s^* - m + Q\left(\omega_n - \omega_{QN}\right) = 0$$

*Up to an undetermined function, the FSC QNMs are:*

$$\omega_n = -\frac{m}{Q} + \frac{1}{Qr_0}(k \pm in\hat{r}_+)(iG + Q\hat{r}_+)$$

*Colleagues are also currently trying to determine the QNMs via the torus two-point function.*

# *dS quotients: Lens spaces*

*Castro, Lashkari, Maloney 2011.*

Consider the Euclidean version of the dS static patch:

$$\frac{ds_E^2}{\ell^2} = dr^2 + \cos^2 r dt_E^2 + \sin^2 r d\phi^2.$$

The Lens space $L(p,q)$ is obtained through the following identification

$$(t_E, \phi) \sim (t_E, \phi) + 2\pi \left( \frac{m}{p}, m\frac{q}{p} + n \right),$$

where $(n, m) \in \mathbb{Z}$. The quotient structure of this space is $S^3/\mathbb{Z}_p$, and the sphere $S^3$ is the special case $L(1,0)$. It seems that this identification is generated by

$$\rho = e^{-2\pi\left(\frac{1}{p}H + i\frac{q}{p}J\right)},$$

where $(H, J)$ are Killing vectors of the unquotiented metric:

$$H = i\partial_t, \qquad J = i\partial_\phi.$$

Our representation theory ansatz tells us to construct

$$\zeta(s) = \prod_{\text{descendants}} \langle 1 - \rho \rangle_{\text{scalar primary weight s}} .$$

# Wilson spools

Castro, Coman, Fliss, Zukowski 2023

A new method of calculating 1-loop determinants in 3D Euclidean de Sitter space.

$$\langle \log Z_{\text{scalar}}[\mathcal{M}] \rangle_{\text{grav}} = \frac{1}{4} \langle \mathbb{W} \rangle_{\text{grav}}$$

The authors call the RHS a Wilson spool. It is a collection of Wilson loops winding around the 3-sphere.

Cool thing: the Wilson spool can be calculated to all orders in the Newton constant, G, due to known results regarding the Chern-Simons formulation of de Sitter. Witten 1988, …

Important for us: The Wilson spool is constructed from "non-standard", nonunitary representations of SU(2).

# Representations of SU(2)

Castro, Sabella-Garnier, Zukowski 2020; Castro, Coman, Fliss, Zukowski 2023

$$[L_3, L_\pm] = \pm L_\pm \quad \text{and} \quad [L_+, L_-] = 2L_3 \qquad c_j = L^2 = L_1^2 + L_2^2 + L_3^2$$

$$c_j \, |j, m\rangle = j(j+1) \, |j, m\rangle$$

| Standard reps | Non-standard reps |
|---|---|
| $L_3^\dagger = L_3 \quad \text{and} \quad L_\pm^\dagger = L_\mp$ | $L_3^\dagger = L_3 \quad \text{and} \quad L_\pm^\dagger = -L_\mp$ |
| Unitary | Non-unitary (!) |
| Finite dimensional | Infinite dimensional |
| $m \leq \lvert j \rvert$ | No restriction |

# Non-standard representations and Selberg

$$Z_\Gamma(s) = \prod_{\text{descendants}} \left\langle 1 - e^{2\pi i \partial_\phi} \right\rangle_{\text{scalar primary of weight } s}$$

For the de Sitter case, we can hope to build TWO different zeta functions, one for the standard rep and one for the non-standard one. We hope to the above prescription to:

1) Construct the 1-loop partition function for Lens spaces using Selberg techniques.

2) Build a Selberg zeta function from the non-standard rep to see if we can learn about them!!

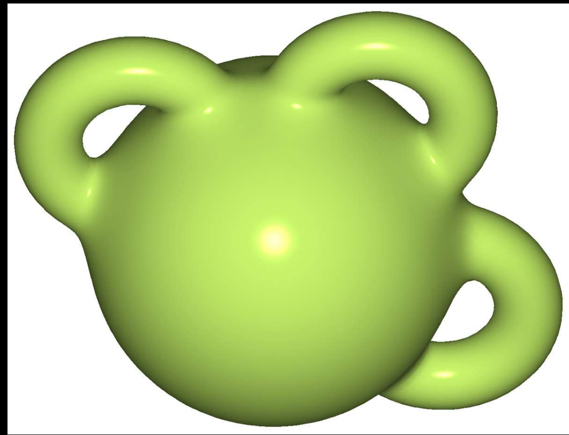
# *Future directions*

Quotient spacetimes are everywhere! Lots of future directions.

1. K-boundary wormholes

2. Warped de Sitter black holes

3. Holographic entanglement



Selberg trace formula as a connection between two differerent methods of calculating functional determinants: The heat kernel method and the quasinormal mode method.

Even more to think about:

- Applications for flat space hologrphy?
- Connection to Seiberg Witten curves?
- L-functions, Langlands program?

Thank you!