2d CFTs, Borcherds products and hyperbolization of affine Lie algebras

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Joint work with Haowu Wang and Brandon Williams, to appear

An interesting coincidence

8 special affine Lie algebras appeared in math and physics around the same time

Corollary (Corollary 7.6). *The following infinite series of pure theta blocks of q-order 1 satisfy the theta block conjecture.*

weight	*root system*	*theta block*
2	A_4	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_c\vartheta_d\vartheta_{a+b}\vartheta_{b+c}\vartheta_{c+d}\vartheta_{a+b+c}\vartheta_{b+c+d}\vartheta_{a+b+c+d}$
	$A_1 \oplus B_3$	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_{b+c}\vartheta_{b+2c+2d}\vartheta_{b+c+d}\vartheta_{b+c+2d}\vartheta_c\vartheta_{c+d}\vartheta_{c+2d}\vartheta_d$
	$A_1 \oplus C_3$	$\eta^{-6}\vartheta_a\vartheta_b\vartheta_{2b+2c+d}\vartheta_{b+c}\vartheta_{b+2c+d}\vartheta_{b+c+d}\vartheta_c\vartheta_{2c+d}\vartheta_{c+d}\vartheta_d$
	$B_2 \oplus G_2$	$\eta^{-6}\vartheta_a\vartheta_{a+b}\vartheta_{a+2b}\vartheta_b\vartheta_c\vartheta_{3c+d}\vartheta_{3c+2d}\vartheta_{2c+d}\vartheta_{c+d}\vartheta_d$
3	$3A_2$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_1+b_1}\vartheta_{b_1}\vartheta_{a_2}\vartheta_{a_2+b_2}\vartheta_{b_2}\vartheta_{a_3}\vartheta_{a_3+b_3}\vartheta_{b_3}$
	$3A_1 \oplus A_3$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_6}\vartheta_{a_4+a_5}\vartheta_{a_5+a_6}\vartheta_{a_4+a_5+a_6}$
	$2A_1 \oplus A_2 \oplus B_2$	$\eta^{-3}\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_3+a_4}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_5+a_6}\vartheta_{a_5+2a_6}\vartheta_{a_6}$
4	$8A_1$	$\vartheta_{a_1}\vartheta_{a_2}\vartheta_{a_3}\vartheta_{a_4}\vartheta_{a_5}\vartheta_{a_6}\vartheta_{a_7}\vartheta_{a_8}$

Figure: Dittmann, Wang, *Theta blocks related to root systems*, 2006.12967.

3. F_{24}: This is a theory of 24 free chiral fermions. One can build an $\mathcal{N} = 1$ superconformal structure by taking a linear combination of cubic Fermi terms, and the allowed choices are classified by semisimple Lie algebras of dimension 24. Each of these generates an affine Kac-Moody algebra, of which there are eight possibilities:

$$(\widehat{su}(2)_2)^{\oplus 8}\,, \quad (\widehat{su}(3)_3)^{\oplus 3}\,, \quad \widehat{su}(4)_4 \oplus (\widehat{su}(2)_2)^{\oplus 3}\,, \quad \widehat{su}(5)_5\,, \quad \widehat{so}(5)_3 \oplus \hat{g}_{2,4}\,,$$
$$\widehat{so}(5)_3 \oplus \widehat{su}(3)_3 \oplus (\widehat{su}(2)_2)^{\oplus 2}\,, \quad \widehat{so}(7)_5 \oplus \widehat{su}(2)_2\,, \quad \widehat{sp}(6)_4 \oplus \widehat{su}(2)_2\,,$$

Figure: Harrison, Paquette, Persson, Volpato, *Fun with F_{24}*, 2009.14710.

Main question

Elliptic Semi-simple Lie algebras – Classical symmetries

Parabolic Affine Kac-Moody algebras – 2d Wess-Zumino-Witten CFTs

Hyperbolic Borcherds-Kac-Moody algebras – Algebra of BPS states (Harvey-Moore 95,96)

Question (Feingold-Frenkel 83, Gritsenko 12, Gritsenko-Wang 19,20...)

What kinds of affine Kac-Moody algebras allow hyperbolization?

Mathematically speaking, this is to classify the reflective Borcherds products $\Phi(\omega, \mathfrak{z}, \tau)$ of singular weight $\frac{1}{2}\mathrm{rk}(L)$ on $2U \oplus L$.

This is a very good class of automorphic forms.

Main results

We give a complete classification of hyperbolization of affine Kac-Moody algebras

Main theorem

There are precisely 81 such affine Kac-Moody algebras:

1. 69 cases associated to Schellenkens' list of $c = 24$ holomorphic CFT/VOAs
2. 8 cases associated to the $c = 12$ holomorphic SCFT/self-dual SVOAs
3. 4 exotic cases $A_{1,16}, A_{2,9}, A_{1,8}^2, A_{1,4}^4$ associated to exceptional modular invariants from nontrivial automorphism of fusion algebras

Example: Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

Recall Jacobi symbol

$$\left(\frac{-4}{m} \right) = \begin{cases} \pm 1, & m \equiv \pm 1 \mod 4, \\ 0, & m \equiv 0 \mod 2, \end{cases}$$

and Jacobi theta function

$$\theta_1(\tau, z) = \sum_{m \in \mathbb{Z}} \left(\frac{-4}{m} \right) q^{m^2/8} r^{m/2}.$$

(Gritsenko-Nikulin 98) defined the Siegel theta constant

$$\Delta_{1/2}(Z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} \left(\frac{-4}{m} \right) \left(\frac{-4}{n} \right) q^{m^2/8} r^{mn/2} s^{n^2/8},$$

and found this is a weight-$1/2$ automorphic form on paramodular group Γ_4^+.

Example: Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

Notably, $\Delta_{1/2}(Z)$ is a reflective Borcherds product of singular weight!

$$\Delta_{1/2}(Z) = \mathbf{B}(\phi) := q^A r^B s^C \prod_{\substack{n,l,m\in\mathbb{Z} \\ (n,l,m)>0}} (1 - q^n r^l s^m)^{f(nm,l)},$$

where $f(-,-)$ are the Fourier coefficients of weight-0 Jacobi form ϕ:

$$\phi(\tau, z) = \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} := \sum_{n,l\in\mathbb{Z}} f(n,l) q^n r^l,$$

and

$$A = \frac{1}{24}\sum_l f(0,l) = \frac{1}{8}, \;\; B = \frac{1}{2}\sum_{l>0} l f(0,l) = \frac{1}{2}, \;\; C = \frac{1}{4}\sum_l l^2 f(0,l) = \frac{1}{8}.$$

Example: Gritsenko-Nikulin's $\Delta_{1/2}(Z)$

- $\Delta_{1/2}(Z)$ is the denominator of a Borcherds-Kac-Moody algebra g with infinite dimensional Cartan matrix.
- This g becomes the hyperbolization of affine Kac-Moody algebra $\mathfrak{g} = A_{1,16}$!
- Notice the weight-0 Jacobi form can be written as

$$\phi(\tau, z) = \frac{\theta_1(\tau, 3z)}{\theta_1(\tau, z)} = \chi_2^{\mathfrak{g}}(\tau, z) + \chi_{14}^{\mathfrak{g}}(\tau, z) - \chi_8^{\mathfrak{g}}(\tau, z).$$

- For $z \to 0$, this reduces to

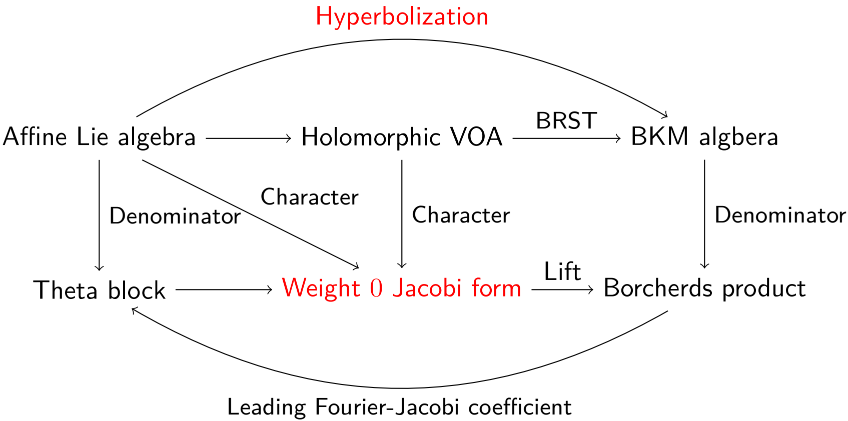
$$3 = \chi_2^{\mathfrak{g}}(\tau) + \chi_{14}^{\mathfrak{g}}(\tau) - \chi_8^{\mathfrak{g}}(\tau),$$

  which is a consequence of the Macdonald identity for $A_{1,2p^2-2}$ with $p = 3$:

$$p = \sum_{j=0}^{p-1} \chi_{2p^2-1-(4j+1)p}^{A_{1,2p^2-2}}(\tau).$$

- The same identity was used in (Moore-Seiberg 88) to construct the E_7 type modular invariant.

Main idea


Hyperbolization
Affine Lie algebra
Holomorphic VOA
BRST
BKM algbera
Denominator
Character
Character
Denominator
Theta block
Weight 0 Jacobi form
Lift
Borcherds product
Leading Fourier-Jacobi coefficient

A necessary condition for $\bigoplus(\mathfrak{g}_i)_{k_i}$

Antisymmetric Reflective Borcherds products for $\bigoplus(\mathfrak{g}_i)_{k_i}$

$$\frac{1}{24}\sum_i \dim(\mathfrak{g}_i) - 1 = C = \frac{h_i^\vee}{k_i}.$$

The central charge $c = 24$. The weight 0 Jacobi form

$$\phi_{\mathrm{Borch}}|_{m_\mathfrak{g}\to 0} = J(\tau) + N = q^{-1} + N + 196884q + \dots .$$

Symmetric Reflective Borcherds products for $\bigoplus(\mathfrak{g}_i)_{k_i}$

$$\frac{1}{24}\sum_i \dim(\mathfrak{g}_i) = C = \frac{h_i^\vee}{k_i}, \qquad k_i > 1.$$

The central charge $c = \frac{24C}{C+1}$. The weight 0 Jacobi form

$$\phi_{\mathrm{Borch}}|_{m_\mathfrak{g}\to 0} = \mathrm{const}.$$

Antisymmetric solutions for $\bigoplus(\mathfrak{g}_i)_{k_i}$, 69 out of 221

C	$\mathfrak{g}$
1/24	$A_{1,48}^3A_{2,72}^2$
1/24	$A_{1,48}^5B_{2,72}$
1/24	$A_{1,48}A_{2,72}G_{2,96}$
1/24	$A_{3,96}B_{2,72}$
1/12	$A_{1,24}^6A_{2,36}$
1/12	$A_{1,24}^4G_{2,48}$
1/12	$A_{1,24}^2B_{2,36}^2$
1/12	$A_{2,36}^2B_{2,36}$
1/12	$A_{1,24}A_{2,36}A_{3,48}$
1/8	$A_{1,16}B_{2,24}G_{2,32}$
1/8	$A_{1,16}A_{2,24}^3$
1/8	$A_{1,16}^2C_{3,32}$
1/8	$A_{1,16}^3A_{2,24}B_{2,24}$
1/8	$A_{1,16}^9$
1/8	$A_{1,16}^4A_{3,32}$
1/8	$A_{1,16}^2B_{3,40}$
1/8	$A_{1,16}A_{4,40}$
1/6	$A_{1,12}^2A_{2,18}G_{2,24}$
1/6	$A_{1,12}^6B_{2,18}$
1/6	$A_{1,12}^4A_{2,18}^2$
1/6	$D_{4,36}$
1/6	$G_{2,24}^2$
1/6	$A_{1,12}A_{3,24}B_{2,18}$
1/6	$A_{2,18}B_{2,18}^2$
1/4	$A_{1,8}^3C_{3,16}$
1/4	$A_{1,8}^{10}$
1/4	$A_{1,8}^2A_{2,12}^3$
1/4	$A_{1,8}^5A_{3,16}$
1/4	$A_{1,8}^2B_{2,12}G_{2,16}$
1/4	$A_{1,8}^3B_{3,20}$
1/4	$A_{3,16}^2$
1/4	$A_{1,8}^2A_{4,20}$
1/4	$A_{1,8}^4A_{2,12}B_{2,12}$
1/4	$A_{2,12}^2G_{2,16}$
1/4	$B_{2,12}^3$
1/3	$A_{2,9}B_{2,9}G_{2,12}$
1/3	$A_{1,6}^8A_{2,9}$
1/3	$A_{1,6}A_{2,9}C_{3,12}$
1/3	$A_{2,9}A_{4,15}$
1/3	$A_{1,6}A_{2,9}B_{3,15}$
1/3	$A_{1,6}^6G_{2,12}$
1/3	$A_{1,6}^2A_{2,9}^2B_{2,9}$
1/3	$A_{1,6}^4B_{2,9}^2$
1/3	$A_{1,6}^3A_{2,9}A_{3,12}$
1/3	$A_{2,9}^4$
1/3	$A_{1,6}A_{3,12}G_{2,12}$
1/2	$A_{2,6}G_{2,8}^2$
1/2	$A_{1,4}^2A_{3,8}^2$
1/2	$A_{1,4}^4A_{2,6}^3$
1/2	$A_{1,4}^4A_{4,10}$
1/2	$A_{1,4}A_{2,6}A_{3,8}B_{2,6}$
1/2	$A_{1,4}^7A_{3,8}$
1/2	$C_{4,10}$
1/2	$A_{1,4}^5B_{3,10}$
1/2	$A_{2,6}^2B_{2,6}^2$
1/2	$A_{3,8}B_{3,10}$
1/2	$A_{1,4}^6A_{2,6}B_{2,6}$
1/2	$A_{1,4}^2A_{2,6}^2G_{2,8}$
1/2	$B_{4,14}$
1/2	$A_{1,4}^{12}$
1/2	$A_{1,4}^4B_{2,6}G_{2,8}$
1/2	$A_{2,6}D_{4,12}$
1/2	$A_{1,4}^5C_{3,8}$
1/2	$A_{1,4}^2B_{2,6}^3$
1/2	$A_{3,8}C_{3,8}$
2/3	$A_{1,3}^4G_{2,6}^2$
2/3	$A_{1,3}^4D_{4,9}$
3/4	$A_{2,4}^4B_{2,4}$
1	$A_{1,2}A_{2,3}^3B_{3,5}$
1	$A_{1,2}^3A_{2,3}B_{2,3}C_{3,4}$
1	$A_{1,2}A_{3,4}B_{2,3}^3$
1	$A_{1,2}A_{4,5}C_{3,4}$
1	$A_{6,7}$
1	$A_{1,2}^4A_{2,3}G_{2,4}^2$
1	$A_{1,2}^4B_{4,7}$
1	$A_{1,2}^{16}$
1	$A_{1,2}A_{5,6}B_{2,3}$
1	$A_{1,2}D_{5,8}$
1	$A_{1,2}^6A_{2,3}^2G_{2,4}$
1	$A_{1,2}^2A_{2,3}B_{2,3}^2G_{2,4}$
1	$A_{1,2}^9B_{3,5}$
1	$B_{2,3}^2D_{4,6}$
1	$A_{1,2}B_{2,3}B_{3,5}G_{2,4}$
1	$A_{2,3}^3A_{4,5}$
1	$A_{1,2}A_{4,5}B_{3,5}$
1	$A_{1,2}^2A_{2,3}A_{4,5}B_{2,3}$
1	$A_{1,2}^2B_{3,5}^2$
1	$A_{1,2}B_{2,3}C_{3,4}G_{2,4}$
1	$A_{1,2}^3A_{2,3}^3A_{3,4}$
1	$A_{1,2}A_{3,4}^3$
1	$A_{1,2}^8A_{2,3}^3$
1	$A_{2,3}^6$
1	$A_{1,2}^{11}A_{3,4}$
1	$A_{1,2}^5A_{2,3}A_{3,4}B_{2,3}$
1	$A_{1,2}^8B_{2,3}G_{2,4}$
1	$A_{1,2}A_{2,3}^3C_{3,4}$
1	$A_{2,3}^3B_{2,3}G_{2,4}$
1	$A_{1,2}^4A_{2,3}D_{4,6}$
1	$A_{1,2}^4A_{3,4}C_{3,4}$
1	$A_{1,2}^3A_{3,4}A_{4,5}$
1	$A_{1,2}^2G_{2,4}^3$
1	$A_{1,2}^2C_{3,4}^2$
1	$A_{1,2}^6A_{3,4}^2$
1	$A_{2,3}A_{3,4}^2B_{2,3}$
1	$A_{1,2}^2A_{2,3}^4B_{2,3}$
1	$A_{1,2}^2B_{3,5}C_{3,4}$
1	$A_{1,2}^8A_{4,5}$
1	$A_{4,5}^2$
1	$A_{1,2}^4A_{2,3}^2B_{2,3}^2$
1	$A_{2,3}B_{2,3}^4$
1	$B_{2,3}^2G_{2,4}^2$
1	$A_{1,2}^9C_{3,4}$
1	$A_{1,2}^4A_{3,4}B_{3,5}$
1	$A_{4,5}B_{2,3}G_{2,4}$
1	$A_{1,2}A_{2,3}^2A_{3,4}G_{2,4}$
1	$A_{1,2}^4C_{4,5}$
1	$A_{1,2}^3A_{2,3}B_{2,3}B_{3,5}$
1	$A_{1,2}^{10}A_{2,3}B_{2,3}$
1	$A_{1,2}^3A_{3,4}B_{2,3}G_{2,4}$
1	$A_{1,2}^2D_{4,6}G_{2,4}$
1	$A_{1,2}^6B_{2,3}^3$
4/3	$C_{3,3}^2G_{2,3}$
4/3	$G_{2,3}^4$

TABLE 3. The 221 solutions of equation (4.1) in the order of increasing C. Those allowing hyperbolization are colored blue (continued on next page).

C	$\mathfrak{g}$
3/2	$A_{2,2}^5B_{2,2}^2$
3/2	$A_{2,2}^4D_{4,4}$
3/2	$B_{2,2}^6$
3/2	$A_{2,2}F_{4,6}$
2	$A_{1,1}^2C_{3,2}D_{5,4}$
2	$A_{1,1}^{10}C_{3,2}^2$
2	$A_{1,1}^5A_{3,2}C_{3,2}^2$
2	$A_{1,1}^9D_{5,4}$
2	$A_{3,2}^2G_{2,2}^3$
2	$A_{1,1}^{14}A_{3,2}^2$
2	$A_{1,1}^{10}G_{2,2}^3$
2	$A_{1,1}^{19}A_{3,2}$
2	$A_{1,1}^5A_{3,2}G_{2,2}^3$
2	$A_{1,1}^3A_{5,3}D_{4,3}$
2	$A_{1,1}^9A_{3,2}^3$
2	$A_{1,1}^{24}$
2	$A_{1,1}^3C_{3,2}D_{4,3}G_{2,2}$
2	$A_{1,1}^7A_{3,2}^2C_{3,2}$
2	$A_{1,1}^3A_{7,4}$
2	$A_{1,1}^2D_{6,5}$
2	$A_{1,1}^5A_{3,2}D_{4,3}G_{2,2}$
2	$A_{1,1}^3C_{3,2}G_{2,2}^3$
2	$A_{1,1}^3C_{3,2}^3$
2	$A_{3,2}^2D_{4,3}G_{2,2}$
2	$A_{1,1}^4A_{3,2}^4$
2	$A_{1,1}^{17}C_{3,2}$
2	$A_{1,1}C_{5,3}G_{2,2}$
2	$A_{3,2}^2C_{3,2}^2$
2	$A_{1,1}^2A_{3,2}^3C_{3,2}$
2	$A_{1,1}^3A_{5,3}G_{2,2}^2$
2	$A_{1,1}^{12}A_{3,2}C_{3,2}$
2	$A_{1,1}^{10}D_{4,3}G_{2,2}$
2	$A_{1,1}^4A_{3,2}D_{5,4}$
5/2	$A_{4,2}^2C_{4,2}$
5/2	$B_{3,2}^4$
3	$A_{2,1}^2A_{8,3}$
3	$A_{2,1}B_{2,1}E_{6,4}$
3	$A_{2,1}^5D_{4,2}^2$
3	$A_{2,1}^2B_{2,1}^8$
3	$A_{2,1}^7B_{2,1}^4$
3	$A_{2,1}^6B_{2,1}^2D_{4,2}$
3	$A_{2,1}^{12}$
3	$A_{2,1}^2D_{4,2}F_{4,3}$
3	$A_{2,1}^3B_{2,1}^2F_{4,3}$
3	$B_{2,1}^4D_{4,2}^2$
3	$A_{2,1}^2A_{5,2}^2B_{2,1}$
3	$A_{2,1}B_{2,1}^6D_{4,2}$
7/2	$B_{4,2}^3$
4	$A_{3,1}C_{7,2}$
4	$A_{3,1}^5D_{5,2}$
4	$A_{3,1}C_{3,1}^5$
4	$A_{3,1}^2D_{5,2}^2$
4	$A_{3,1}A_{7,2}C_{3,1}^2$
4	$C_{3,1}^2E_{6,3}$
4	$A_{3,1}^8$
4	$A_{3,1}C_{3,1}G_{2,1}^6$
4	$A_{3,1}A_{7,2}G_{2,1}^3$
4	$A_{3,1}D_{7,3}G_{2,1}$
4	$A_{3,1}C_{3,1}^3G_{2,1}^3$
4	$E_{6,3}G_{2,1}^3$
9/2	$A_{8,2}F_{4,2}$
5	$A_{4,1}B_{3,1}^4C_{4,1}$
5	$A_{4,1}^6$
5	$A_{4,1}^3C_{4,1}^2$
5	$A_{4,1}A_{9,2}B_{3,1}$
5	$C_{4,1}^4$
5	$B_{3,1}^2C_{4,1}D_{6,2}$
11/2	$B_{6,2}^2$
6	$D_{4,1}^6$
6	$A_{5,1}C_{5,1}E_{6,2}$
6	$A_{5,1}E_{7,3}$
6	$A_{5,1}^4D_{4,1}$
7	$B_{4,1}^2D_{8,2}$
7	$B_{4,1}C_{6,1}^2$
7	$A_{6,1}^4$
7	$A_{6,1}B_{4,1}^4$
8	$A_{7,1}D_{9,2}$
8	$A_{7,1}^2D_{5,1}^2$
9	$C_{8,1}F_{4,1}^2$
9	$B_{5,1}E_{7,2}F_{4,1}$
9	$A_{8,1}^3$
10	$D_{6,1}^4$
10	$A_{9,1}^2D_{6,1}$
11	$B_{6,1}C_{10,1}$
23/2	$B_{12,2}$
12	$E_{6,1}^4$
12	$A_{11,1}D_{7,1}E_{6,1}$
13	$A_{12,1}^2$
14	$D_{8,1}^3$
15	$B_{8,1}E_{8,2}$
16	$A_{15,1}D_{9,1}$
18	$D_{10,1}E_{7,1}^2$
18	$A_{17,1}E_{7,1}$
22	$D_{12,1}^2$
25	$A_{24,1}$
30	$E_{8,1}^3$
30	$D_{16,1}E_{8,1}$
46	$D_{24,1}$

TABLE 3. (continued).

Schellenkens' list

In 1993, Schellenkens classified holomorphic CFTs of $c = 24$. There are in total 71 of them:

1. Monster CFT, $N = 0$, Monster Lie superalgebra
2. Leech CFT, $N = 24$, fake Monster Lie superalgebra
3. 69 cases with affine Kac-Moody structures, $N \geq 36$

After laborious works in lattice theory and Borcherds products for all 221 solutions, we prove surprisingly

Main results

The affine Lie algebras allowing antisymmetric hyperbolization are one to one corresponding to the 69 affine structures in Schellenkens' list and $\phi_{\mathrm{Borch}} = \chi_V$.

Symmetric solutions for $\bigoplus(\mathfrak{g}_i)_{k_i}$, 12 out of 17

C	$\mathfrak{g}$
1/8	$A_{1,16}$
1/4	$A_{1,8}^2$
1/3	$A_{2,9}$
1/2	$A_{1,4}^4$
3/4	$A_{2,4}B_{2,4}$
1	$A_{1,2}^8$

C	$\mathfrak{g}$
1	$A_{2,3}^3$
1	$A_{4,5}$
1	$A_{3,4}A_{1,2}^3$
1	$B_{2,3}G_{2,4}$
1	$B_{2,3}A_{2,3}A_{1,2}^2$
1	$B_{3,5}A_{1,2}$

C	$\mathfrak{g}$
1	$C_{3,4}A_{1,2}$
3/2	$A_{2,2}D_{4,4}$
3/2	$A_{2,2}^2B_{2,2}^2$
5/2	$A_{4,2}C_{4,2}$
7/2	$A_{6,2}B_{4,2}$

$C = 1$ Eight affine structures in F_{24} SCFT of $c = 12$

$C < 1$ Four exotic cases related to exceptional modular invariants

A simple example: $B_{12,2}$, Schellekens' list No.57

The holomorphic CFT character can be expressed by affine characters as

$$\chi_V = \chi_{0,0}^{(B_{12})_2} + \chi_{w_1+w_{12},2}^{(B_{12})_2} + \chi_{w_{10},3}^{(B_{12})_2} + \chi_{w_5,2}^{(B_{12})_2}.$$

Decompose the reps into Weyl orbits with norm defined by $(,)_{B_{12}}/2$. Then $\chi_V = q^{-1} + (O_{w_2,1} + O_{w_1,\frac{1}{2}} + 12) + \sum_{i=1}^{\infty} c_i q^i$. We calculate

$$\begin{aligned} c_1 = & O_{2w_2,4} + O_{w_1+w_3,3} + O_{w_5,\frac{5}{2}} + O_{w_1+w_{12},\frac{5}{2}} + O_{w_1+w_2,\frac{5}{2}} + 4O_{w_4,2} \\ & + 12O_{2w_1,2} + 12O_{w_3,\frac{3}{2}} + 12O_{w_{12},\frac{3}{2}} + 44O_{w_2,1} + 90O_{w_1,\frac{1}{2}} + 300. \end{aligned}$$

Clearly all orbits in c_1 with norm > 2 have coefficients 1.

A simple example: $B_{12,2}$, Schellekens' list No.57

The holomorphic CFT character can be expressed by affine characters as

$$\chi_V = \chi_{0,0}^{(B_{12})_2} + \chi_{w_1+w_{12},2}^{(B_{12})_2} + \chi_{w_{10},3}^{(B_{12})_2} + \chi_{w_5,2}^{(B_{12})_2}.$$

Decompose the reps into Weyl orbits with norm defined by $(,)_{B_{12}}/2$. Then $\chi_V = q^{-1} + (O_{w_2,1} + O_{w_1,\frac{1}{2}} + 12) + \sum_{i=1}^{\infty} c_i q^i$. We calculate

$$\begin{aligned} c_1 = & O_{2w_2,4} + O_{w_1+w_3,3} + O_{w_5,\frac{5}{2}} + O_{w_1+w_{12},\frac{5}{2}} + O_{w_1+w_2,\frac{5}{2}} + 4O_{w_4,2} \\ & + 12O_{2w_1,2} + 12O_{w_3,\frac{3}{2}} + 12O_{w_{12},\frac{3}{2}} + 44O_{w_2,1} + 90O_{w_1,\frac{1}{2}} + 300. \end{aligned}$$

Clearly all orbits in c_1 with norm > 2 have coefficients 1.

$$\begin{aligned} c_2 = & O_{w_{10},5} + O_{2w_0+w_1,5} + O_{w_2+w_4,5} + O_{w_9,\frac{9}{2}} + O_{3w_1,\frac{9}{2}} + O_{w_2+w_3,\frac{9}{2}} + O_{2w_1+w_{12},\frac{9}{2}} \\ & + O_{w_3+w_{12},\frac{9}{2}} + O_{w_1+w_6,\frac{9}{2}} + 4O_{w_8,4} + 12O_{2w_2,4} + 4O_{w_1+w_5,4} + 12O_{w_7,\frac{7}{2}} \\ & + 12O_{w_2+w_{12},\frac{7}{2}} + 12O_{w_1+w_4,\frac{7}{2}} + 32O_{w_6,3} + 44O_{w_1+w_3,3} + 90O_{w_5,\frac{5}{2}} \\ & + 90O_{w_1+w_{12},\frac{5}{2}} + 90O_{w_1+w_2,\frac{5}{2}} + 224O_{w_4,2} + 288O_{2w_1,2} \\ & + 520O_{w_{12},\frac{3}{2}} + 520O_{w_3,\frac{3}{2}} + 1242O_{w_2,1} + 2535O_{w_1,\frac{1}{2}} + 5792. \end{aligned}$$

All orbits with norm > 4 in c_2 have coefficients 1. This implies singular weight!

2d holomorphic SCFTs with $c = 12$

2d holomorphic SCFTs/self-dual vertex operator superalgebras with $c = 12$ only have three types (Creutzig-Duncan-Riedler 18)

1. supersymmetric $E_{8,1}$
2. Conway SCFT
3. 24 free chiral fermions F_{24}

F_{24} allows 8 affine Kac-Moody structures (Harrison-Paquette-Persson-Volpato 20)

$$A_{1,2}^8,\ A_{2,3}^3,\ A_{4,5},\ A_{3,4}A_{1,2}^3,\ B_{2,3}G_{2,4},\ B_{2,3}A_{2,3}A_{1,2}^2,\ B_{3,5}A_{1,2},\ C_{3,4}A_{1,2}.$$

These 8 affine Kac-Moody algebras can be comformally embedded in $SO(24)_1$.

2d holomorphic SCFTs with $c = 12$

The fermionic characters of the eight F_{24} SCFTs are computed as

$$\chi = \eta^{-12}\theta_i^{r/2}\prod_{\alpha\in\Delta_+}\theta_i(z_\alpha),\quad i=3,4,2 \text{ for } \mathrm{NS},\widetilde{\mathrm{NS}},\mathrm{R}.$$

The $\widetilde{\mathrm{R}}$ sectors have $\chi = 0$. Then the fermionic partition function is

$$Z_{\mathrm{F}} = \frac{1}{2}(|\chi_{\mathrm{NS}}|^2 + |\chi_{\widetilde{\mathrm{NS}}}|^2 + |\chi_{\mathrm{R}}|^2).$$

The input Jacobi form of Borcherds product is given by

$$\phi_{\mathrm{Borch}} = \chi_{\mathrm{NS}} - \chi_{\widetilde{\mathrm{NS}}} - \chi_{\mathrm{R}}.$$

The 4 exotic CFTs

In math literature, there are four more known reflective Borcherds products of singular weights. In affine Lie algebra language, they are

$\mathfrak{g}$	$A_{1,16}$	$A_{1,8}^2$	$A_{1,4}^4$	$A_{2,9}$
C	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$
c	$\frac{8}{3}$	$\frac{24}{5}$	8	6
rk	1	2	4	2
dim	3	6	12	8
ref	Gritsenko-Nikulin 98	Grit 19	Grit 18	Gritsenko-Skoruppa-Zagier 19

Question

Is there any physical meaning for these four Borcherds products?

The 4 exotic CFTs

Answer

Yes! They are related to some very peculiar exceptional modular invariants that come from the nontrivial automorphism of the fusion algebra of the simple current extension.

Such peculiarity of $A_{1,16}$ and $A_{2,9}$ was first noticed by (Moore-Seiberg 88)! later for $A_{1,8}^2$ by (Verstegen 90) and for $A_{1,4}^4$ by (Gannon 94).

The nontrivial automorphism of the fusion algebra happens rarely. It can be proved for $A_{1,k}$ this only happens at $k = 16$, while for $A_{2,k}$ only at $k = 9$.

Example: $A_{1,16}$

Affine A_1 has an ADE classification of modular invariants. The D_{10} modular invariant of $A_{1,16}$, – a simple current extended modular invariant:

$$\begin{aligned} Z_{D_{10}} &= |\chi_0+\chi_{16}|^2+|\chi_2+\chi_{14}|^2+|\chi_4+\chi_{12}|^2+|\chi_6+\chi_{10}|^2+2|\chi_8|^2 \\ &= |\phi_0|^2+|\phi_1|^2+|\phi_2|^2+|\phi_3|^2+|\phi_4|^2+|\phi_4'|^2. \end{aligned}$$

The S-matrix for the six extended fields $\phi_{0,1,2,3,4,4'}$ is

$$\frac{1}{3}\begin{pmatrix} 2\sin\left(\frac{\pi}{18}\right) & 1 & 2\cos\left(\frac{2\pi}{9}\right) & 2\cos\left(\frac{\pi}{9}\right) & 1 & 1 \\ 1 & 2 & 1 & -1 & -1 & -1 \\ 2\cos\left(\frac{2\pi}{9}\right) & 1 & -2\cos\left(\frac{\pi}{9}\right) & -2\sin\left(\frac{\pi}{18}\right) & 1 & 1 \\ 2\cos\left(\frac{\pi}{9}\right) & -1 & -2\sin\left(\frac{\pi}{18}\right) & 2\cos\left(\frac{2\pi}{9}\right) & -1 & -1 \\ 1 & -1 & 1 & -1 & 2 & -1 \\ 1 & -1 & 1 & -1 & -1 & 2 \end{pmatrix}.$$

(Moore-Seiberg 88) observed a nontrivial automorphism $\omega : \phi_1 \leftrightarrow \phi_4$ for the fusion algebra! This only happens at level $k = 16$!

E_7 modular invariant

From such automorphism, (Moore-Seiberg 88) obtained the E_7 modular invariant:

$$\begin{aligned} Z_{E_7} &= Z_{D_{10}}\Big|_{\omega \text{ in holomorphic sector}} \\ &= |\phi_0|^2 + \phi_1\bar{\phi}_4 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4'|^2 + \phi_4\bar{\phi}_1 \\ &= |\chi_0+\chi_{16}|^2 + |\chi_4+\chi_{12}|^2 + |\chi_6+\chi_{10}|^2 + |\chi_8|^2 \\ &\quad + ((\chi_2+\chi_{14})\bar{\chi}_8 + c.c.) \\ &= Z_{D_{10}} - |\phi_{A_1}|^2. \end{aligned}$$

The weight 0 Jacobi form for Borcherds product

$$\phi_{A_1} = \frac{\theta_1(3z)}{\theta_1(z)} = \chi^{A_{1,16}}_{2,\frac{1}{9}} + \chi^{A_{1,16}}_{14,\frac{28}{9}} - \chi^{A_{1,16}}_{8,\frac{10}{9}}.$$

Thank you!