Classifying large N limits of multiscalar theories by algebra

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Based on recent work and 2303.13884 with Bo Sundborg

Introduction

- Earlier work: RG flow for multiscalar theories in 4D can be described by non-associative algebras [2303.13884 with Bo Sundborg]
- Non-associative algebras long studied in mathematics [Markus 1960] [Krasnov 2023]

- Main result of present work on multiscalar theory
  - For large N, leading order RG flows separate into 1d and 2d flows via a decomposition of the algebra into simple ideals
  - The algebra lets us identify large N limits via scaling arguments
  - Example model: multiscalar theory with SU(N)xO(M) symmetry

Algebraic description at one-loop

- Multiscalar theory with massless scalars and quartic interactions in 4D

$$\mathcal{L}_{\text{int}} = -\frac{1}{4!}\lambda_{ABCD}\phi_A\phi_B\phi_C\phi_D$$

- One-loop beta function

$$\frac{d}{dt}\lambda_{ABCD} = \mu\frac{d}{d\mu}\lambda_{ABCD} = \beta_{ABCD} = \frac{1}{(4\pi)^2}\Lambda^2_{ABCD}$$

$$\Lambda^2_{ABCD} = \frac{1}{8}\sum_{\text{perms}}\lambda_{ABEF}\lambda_{EFCD}$$

Algebraic description at one-loop

- Beta function is quadratic in the couplings

$$\beta_\lambda = \frac{1}{(4\pi)^2} P_2(\lambda)$$

- Gives rise to a product [Michel, Radicati 1971]
  - Commutative
  - Not generally associative

$$\lambda \diamond \kappa \equiv \frac{1}{2}(P_2(\lambda+\kappa) - P_2(\lambda) - P_2(\kappa))$$
$$= \frac{(4\pi)^2}{2}(\beta_{\lambda+\kappa} - \beta_\lambda - \beta_\kappa),$$

Algebraic description at one-loop

- Introduce a basis closed under RG flow $\lambda_{ABCD} = \lambda_k e^k_{ABCD}$

$$\beta_{ABCD} = \beta_k e^k_{ABCD} \qquad \beta_k = \frac{1}{(4\pi)^2}\lambda_m \lambda_n C_k^{mn}$$

$$(e^m \diamond e^n)_{ABCD} \equiv \frac{1}{8}\sum_{\text{perms}} e^m_{ABEF} e^n_{EFCD} = C_k^{mn} e^k_{ABCD}$$

$$\begin{pmatrix} e^1 \diamond e^1 & \dots & e^1 \diamond e^K \\ \vdots & \ddots & \vdots \\ e^K \diamond e^1 & \dots & e^K \diamond e^K \end{pmatrix}$$

Multiscalar model SU(N)xO(M)

- SU(N)xO(M) model $A = \bar{a}\bar{A}$

$\bar{A} = 1, \ldots, N^2 - 1$ is the $SU(N)$-index

$\bar{a} = 1, \ldots, M$ is the scalar multiplet index

$$\Phi_{\bar{a}} = \phi_A T_{\bar{A}} = \phi_{\bar{a}\bar{A}} T_{\bar{A}}$$

- Choose a basis

e^k_{ABCD}

$k = \{1s, 1t, 2s, 2t\}$

$$\frac{1}{4!} e^{1s}_{ABCD} \phi_A \phi_B \phi_C \phi_D = \frac{1}{2} \operatorname{Tr} \Phi_{\bar{a}} \Phi_{\bar{a}} \Phi_{\bar{b}} \Phi_{\bar{b}}$$

$$\frac{1}{4!} e^{1t}_{ABCD} \phi_A \phi_B \phi_C \phi_D = \frac{1}{4} \operatorname{Tr} \Phi_{\bar{a}} \Phi_{\bar{b}} \Phi_{\bar{a}} \Phi_{\bar{b}}$$

$$\frac{1}{4!} e^{2s}_{ABCD} \phi_A \phi_B \phi_C \phi_D = \frac{1}{2} \operatorname{Tr} \Phi_{\bar{a}} \Phi_{\bar{a}} \operatorname{Tr} \Phi_{\bar{b}} \Phi_{\bar{b}}$$

$$\frac{1}{4!} e^{2t}_{ABCD} \phi_A \phi_B \phi_C \phi_D = \operatorname{Tr} \Phi_{\bar{a}} \Phi_{\bar{b}} \operatorname{Tr} \Phi_{\bar{a}} \Phi_{\bar{b}}.$$

Multiscalar model SU(N)xO(M)

- Algebra (not large N)

$$\begin{pmatrix} e^{1s}\diamond e^{1s} & e^{1s}\diamond e^{1t} \\ e^{1t}\diamond e^{1s} & e^{1t}\diamond e^{1t} \end{pmatrix} = \begin{pmatrix} \frac{(M+3)}{2}e^{2s} + \frac{1}{2}e^{2t} + \frac{N(M+3)}{2}e^{1s} & \frac{1}{2}e^{2s} + \frac{1}{2}e^{2t} + \frac{N}{2}e^{1s} + Ne^{1t} \\ \frac{1}{2}e^{2s} + \frac{1}{2}e^{2t} + \frac{N}{2}e^{1s} + Ne^{1t} & \frac{(M+2)}{8}e^{2t} + \frac{N}{2}e^{1s} \end{pmatrix}$$

$$\begin{pmatrix} e^{2s}\diamond e^{2s} & e^{2s}\diamond e^{2t} \\ e^{2t}\diamond e^{2s} & e^{2t}\diamond e^{2t} \end{pmatrix} = \begin{pmatrix} (M(N^2-1)+8)e^{2s} & 2(M+N^2)e^{2s} + 6e^{2t} \\ 2(M+N^2)e^{2s} + 6e^{2t} & (2M+2N^2+6)e^{2t} + 12e^{2s} \end{pmatrix}$$

$$\begin{pmatrix} e^{1s}\diamond e^{2s} & e^{1s}\diamond e^{2t} \\ e^{1t}\diamond e^{2s} & e^{1t}\diamond e^{2t} \end{pmatrix} = \begin{pmatrix} (M+1)Ne^{2s} + 6e^{1s} & 2(M+3)e^{1s} + 2Ne^{2s} + Ne^{2t} + 8e^{1t} \\ Ne^{2s} + 6e^{1t} & 2(M+1)e^{1t} + Ne^{2t} + 4e^{1s} \end{pmatrix}$$

Rescaling the basis

- Rescale to take large N limit $\lambda_k e^k = \Lambda_k E^k$

$$e^k = N^{n(k)} M^{m(k)} E^k$$

$$\boxed{M = v(a) N^a}$$

$$e^k = N^{n(k)+am(k)} v(a)^{m(k)} E^k$$

- Demand finite elements for large N
  - Constrains p(k) and *a*

$$p(k) = n(k) + am(k)$$

Rescaling the basis

- Constraint: 0≤a≤2
- Most general rescaling

$$\lambda_{1s} = \frac{\Lambda_{1s}}{MN} = \frac{\lambda_{1S}}{MN}$$
$$\lambda_{1t} = \frac{\Lambda_{1t}}{\sqrt{MN}} = \frac{\lambda_{1T}}{\sqrt{MN}}$$
$$\lambda_{2s} = \frac{\Lambda_{2s}}{MN^2} = \frac{\lambda_{2S}}{MN^2}$$
$$\lambda_{2t} = \frac{\Lambda_{2t}}{N^2} = \frac{\lambda_{2T}}{N^2}.$$

Rescaling the basis

- 3 large N (and M) limits
- Case $a=2$: multi-matrix limit
- Case $0<a<2$: intermediate limit
- Case $a=0$: regular ’t Hooft limit

Properties of the algebra

- Case a=2
  - Free parameter $v=M/N^2$ appears

$\diamond$	e^{1S}	e^{1T}	e^{2S}	e^{2T}
e^{1S}	$\frac{1}{2}e^{1S}+\frac{1}{2}e^{2S}$	0	e^{2S}	$2ve^{1S}+2e^{2S}$
e^{1T}	0	$\frac{1}{2}e^{1S}+\frac{1}{8}e^{2T}$	0	$2ve^{1T}$
e^{2S}	e^{2S}	0	e^{2S}	$(2+2v)e^{2S}$
e^{2T}	$2ve^{1S}+2e^{2S}$	$2ve^{1T}$	$(2+2v)e^{2S}$	$12ve^{2S}+(2+2v)e^{2T}$

Properties of the algebra

- Case 0<a<2
  - Free parameter v(a) from $M=v(a)N^a$ does not appear
  - e^{1T} not generated → $\beta_{1T} = 0$
  - Limit v→0 for a=2 gives this case
  - Limit M→∞ for a=0 gives this case
  - Associative

$\diamond$	e^{1S}	e^{1T}	e^{2S}	e^{2T}
e^{1S}	$\frac{1}{2}e^{1S} + \frac{1}{2}e^{2S}$	0	e^{2S}	$2e^{2S}$
e^{1T}	0	$\frac{1}{2}e^{1S} + \frac{1}{8}e^{2T}$	0	0
e^{2S}	e^{2S}	0	e^{2S}	$2e^{2S}$
e^{2T}	$2e^{2S}$	0	$2e^{2S}$	$2e^{2T}$

Properties of the algebra

- Case a=0
  - Free parameter M=constant appears

$\diamond$	e^{1S}	e^{1T}	e^{2S}	e^{2T}
e^{1S}	$(\frac{1}{2}+\frac{3}{2M})e^{1S}+(\frac{1}{2}+\frac{3}{2M})e^{2S}+\frac{1}{2M^2}e^{2T}$	$\frac{1}{2\sqrt{M}}e^{1S}+\frac{1}{M}e^{1T}+\frac{1}{2\sqrt{M}}e^{2S}+\frac{1}{2M^{3/2}}e^{2T}$	$(1+\frac{1}{M})e^{2S}$	$2e^{2S}+\frac{1}{M}e^{2T}$
e^{1T}	$\frac{1}{2\sqrt{M}}e^{1S}+\frac{1}{M}e^{1T}+\frac{1}{2\sqrt{M}}e^{2S}+\frac{1}{2M^{3/2}}e^{2T}$	$\frac{1}{2}e^{1S}+(\frac{1}{8}+\frac{1}{4M})e^{2T}$	$\frac{1}{\sqrt{M}}e^{2S}$	$\frac{1}{\sqrt{M}}e^{2T}$
e^{2S}	$(1+\frac{1}{M})e^{2S}$	$\frac{1}{\sqrt{M}}e^{2S}$	e^{2S}	$2e^{2S}$
e^{2T}	$2e^{2S}+\frac{1}{M}e^{2T}$	$\frac{1}{\sqrt{M}}e^{2T}$	$2e^{2S}$	$2e^{2T}$

Properties of the algebra

- Notation: $A = \{e^{1S}, e^{1T}, e^{2S}, e^{2T}\}$
- Subalgebras: closed subspace of the algebra → renormalizable subtheory
  - Shows which couplings induce other couplings
  - Ex: case a=2 has subalgbra $\{e^{1S}, e^{2S}\}$

$\diamond$	e^{1S}	e^{1T}	e^{2S}	e^{2T}
e^{1S}	$\frac{1}{2}e^{1S} + \frac{1}{2}e^{2S}$	0	e^{2S}	$2ve^{1S} + 2e^{2S}$
e^{1T}	0	$\frac{1}{2}e^{1S} + \frac{1}{8}e^{2T}$	0	$2ve^{1T}$
e^{2S}	e^{2S}	0	e^{2S}	$(2+2v)e^{2S}$
e^{2T}	$2ve^{1S} + 2e^{2S}$	$2ve^{1T}$	$(2+2v)e^{2S}$	$12ve^{2S} + (2+2v)e^{2T}$

Properties of the algebra

- Ideals (I): subalgebra with the requirement that the product of any element of the algebra with an element of an ideal belongs to the ideal
  - Ex: case a=2 has ideal {e^{2S}}

$\diamond$	e^{1S}	e^{1T}	e^{2S}	e^{2T}
e^{1S}	$\frac{1}{2}e^{1S} + \frac{1}{2}e^{2S}$	0	e^{2S}	$2ve^{1S} + 2e^{2S}$
e^{1T}	0	$\frac{1}{2}e^{1S} + \frac{1}{8}e^{2T}$	0	$2ve^{1T}$
e^{2S}	e^{2S}	0	e^{2S}	$(2+2v)e^{2S}$
e^{2T}	$2ve^{1S} + 2e^{2S}$	$2ve^{1T}$	$(2+2v)e^{2S}$	$12ve^{2S} + (2+2v)e^{2T}$

Properties of the algebra

- Quotient algbras of ideals (A/I)
  - Ideal modded out
  - RG equations for couplings in the quotient algebra form a closed dynamical system, independent of the couplings of the ideal
  - Considering all the ideals/quotient algebras → natural order to solve RG equations

Properties of the algebra

- Symmetry-respecting basis
- Another basis?

	a=0	0<a<2	a=2
Subalgebras	$\{e^{2S}\}$, $\{e^{2T}\}$, $\{e^{2S}, e^{2T}\}$, $\{e^{1S}, e^{2S}, e^{2T}\}$	$\{e^{2S}\}$, $\{e^{2T}\}$, $\{e^{1S}, e^{2S}\}$, $\{e^{2S}, e^{2T}\}$, $\{e^{1S}, e^{2S}, e^{2T}\}$	$\{e^{2S}\}$, $\{e^{1S}, e^{2S}\}$, $\{e^{2S}, e^{2T}\}$, $\{e^{1S}, e^{2S}, e^{2T}\}$
Ideals	$\{e^{2S}\}$, $\{e^{2S}, e^{2T}\}$	$\{e^{2S}\}$, $\{e^{1S}, e^{2S}\}$, $\{e^{2S}, e^{2T}\}$, $\{e^{1S}, e^{2S}, e^{2T}\}$	$\{e^{2S}\}$, $\{e^{1S}, e^{2S}\}$
Quotient algebras	$A/\{e^{2S}\}$, $A/\{e^{2S}, e^{2T}\}$	$A/\{e^{2S}\}$, $A/\{e^{1S}, e^{2S}\}$, $A/\{e^{2S}, e^{2T}\}$, $A/\{e^{1S}, e^{2S}, e^{2T}\}$	$A/\{e^{2S}\}$, $A/\{e^{1S}, e^{2S}\}$

- Algebra can be decomposed into a direct sum of independent simple ideals, each with their own independent RG equations
  - Simple ideal: no non-trivial sub-ideals
  - Given a positive definite bilinear form of the non-associative algebra

Properties of the algebra

- Trace-form: symmetric bilinear form (x, y)

$$(x \diamond y, z) = (x, y \diamond z) \qquad x, y, z \in A$$

- For our algebra
  - Positive definite → non-degenerate

$$(u, v) = u_{ABCD} v_{ABCD} = (v, u)$$

Properties of the algebra

- Non-degenerate trace-form → orthogonal complement of ideal is an ideal
  - Orthogonal complement of I: $I_\perp = \{y|(x,y) = 0 \;\forall\; x \in I\}$
- For a positive definite bilinear trace-form → $A = I \oplus I_\perp$
  - Start with a simple ideal S: $A = S \oplus S_\perp$
  - Repeat decomposition for $S_\perp$
  - Full decomposition into simple ideals $A = S^1 \oplus \cdots \oplus S^k$
- Isomorphism $I_\perp \simeq A/I$
- Basis from decomposition → independent RG eqs for each simple ideal

Decompositions at large N

- Special elements: idempotents & nilpotents
  - Idempotent $\mathbf{c}^2 \equiv \mathbf{c} \diamond \mathbf{c} = \mathbf{c}$
  - Nilpotent $\mathbf{n}^2 \equiv \mathbf{n} \diamond \mathbf{n} = 0$
  - Peirce decomposition [Krasnov 2023] →divide the flow into sectors
  - Each 1d ideal is spanned by an idempotent or nilpotent
  - Appear in the RG flow

- Idempotent: a coupling that reproduces itself when squared → RG flows in 1d linear subspaces of couplings

Decompositions at large N

- Case a=2 ($v=M/N^2$) $$A = S^{2S} \oplus (S_{\perp}^{2S} \cap I^S) \oplus S^{OS}$$
  1d 1d 2d
  - 3 closed dynamical systems
- Case a=0 (M constant) $$A = S^{2S} \oplus (S_{\perp}^{2S} \cap I^2) \oplus S^{O2}$$
  1d 1d 2d
  - 3 closed dynamical systems

$$I^S = \{e^{1S}, e^{2S}\}$$
$$S^{OS} = I_{\perp}^S$$

$$S^{2S} = \{e^{2S}\}$$

$$I^2 = \{e^{2S}, e^{2T}\}$$
$$S^{O2} = I_{\perp}^2$$

Decompositions at large N

- Case 0<a<2 $A = S^t \oplus S^{2S} \oplus (S_\perp^{2S} \cap I^2) \oplus (S_\perp^{2S} \cap I^S)$
  - 4 closed dynamical systems
  - 3 of them spanned by idempotents
  - 1 spanned by a nilpotent S^t → $\beta_{1T} = 0$

$$I^{Ot} = \{e^{1S}, e^{2S}, e^{2T}\}$$
$$S^t = I_\perp^{Ot}$$
$$I^2 = \{e^{2S}, e^{2T}\}$$
$$I^S = \{e^{1S}, e^{2S}\}$$
$$S^{OS} = I_\perp^S$$
$$S^{O2} = I_\perp^2$$
$$S^{2S} = \{e^{2S}\}$$

RG flow

- Case a=2 ($v=M/N^2$)
  - Only trivial fixed point
  - Study 2d simple ideal S^{OS}: 3 idempotents

$$\beta_{1S} = \frac{1}{32\pi^2}(\lambda_{1S}^2 + (1-v)\lambda_{1T}^2 + 8v\lambda_{1S}\lambda_{2T})$$

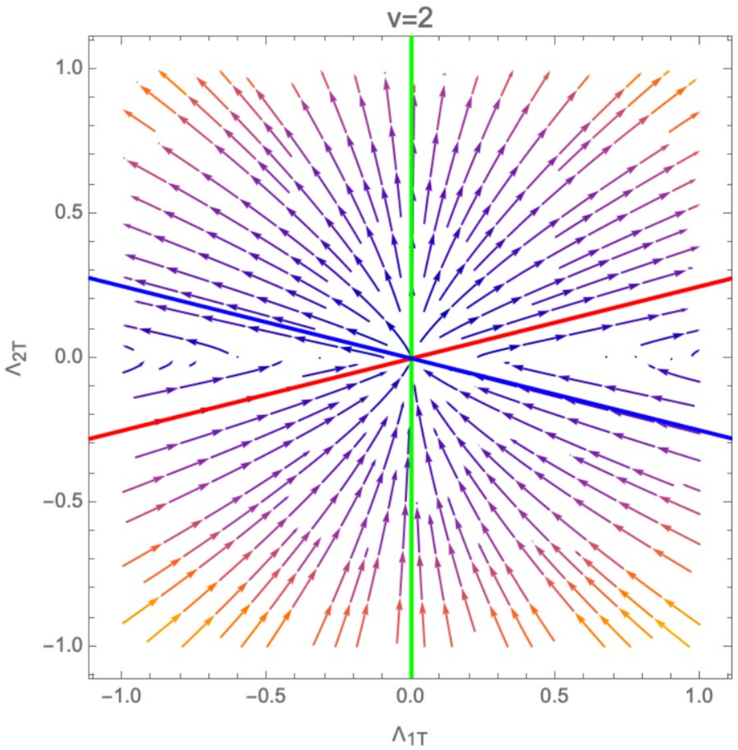
$$\beta_{1T} = \frac{1}{4\pi^2} v\lambda_{1T}\lambda_{2T}$$

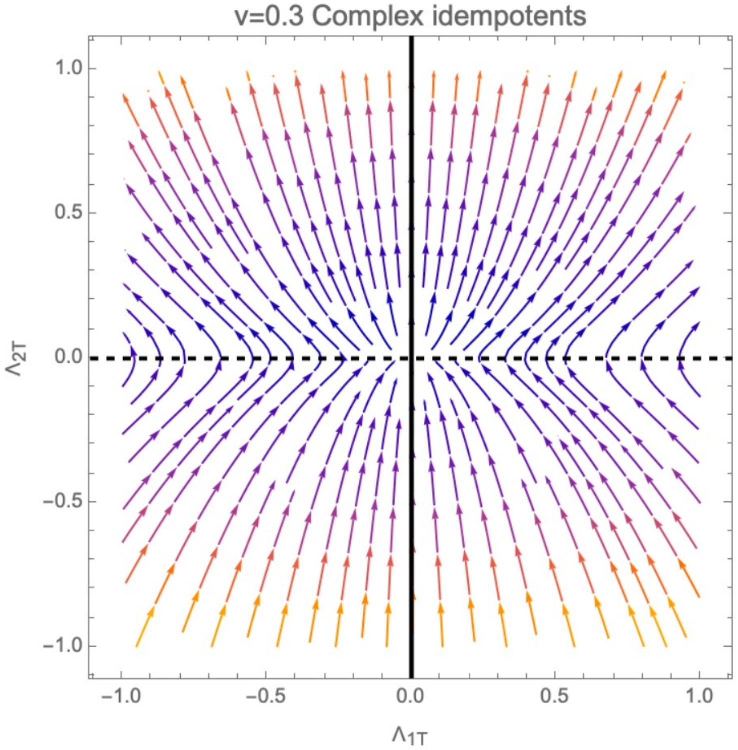
$$\beta_{2S} = \frac{1}{64\pi^2}(2\lambda_{1S}^2 + 8\lambda_{1S}(\lambda_{2S} + 2\lambda_{2T}) + 4\lambda_{2S}(\lambda_{2S} + 4\lambda_{2T}) + v(\lambda_{1T}^2 + 16\lambda_{2T}(\lambda_{2S} + 3\lambda_{2T})))$$

$$\beta_{2T} = \frac{1}{128\pi^2}(\lambda_{1T}^2 + 16(1+v)\lambda_{2T}^2).$$

RG flow

- $(\lambda_{1T}, \lambda_{2T})$-space $\quad S^{OS} \cong A/I^S \qquad I^S = \{e^{1S}, e^{2S}\} \qquad S^{OS} = I^S_\perp$
- Idempotents all real for v≥1





RG flow

- Case 0<a<2 ($M=v(a)N^a$)
  - Nilpotent ideal → vanishing beta function
  - Solution with complex λ_{1T} (tensor models [Benedetti,Gurau,Harribey,2019])

$$\begin{aligned}
\beta_{1S} =& \frac{1}{32\pi^2}(\lambda_{1S}^2 + \lambda_{1T}^2) \\
\beta_{1T} =& 0 \\
\beta_{2S} =& \frac{1}{32\pi^2}(\lambda_{1S}^2 + 4\lambda_{1S}\lambda_{2S} + 2\lambda_{2S}^2 + 8(\lambda_{1S} + \lambda_{2S})\lambda_{2T}) \\
\beta_{2T} =& \frac{1}{128\pi^2}(\lambda_{1T}^2 + 16\lambda_{2T}^2).
\end{aligned}$$

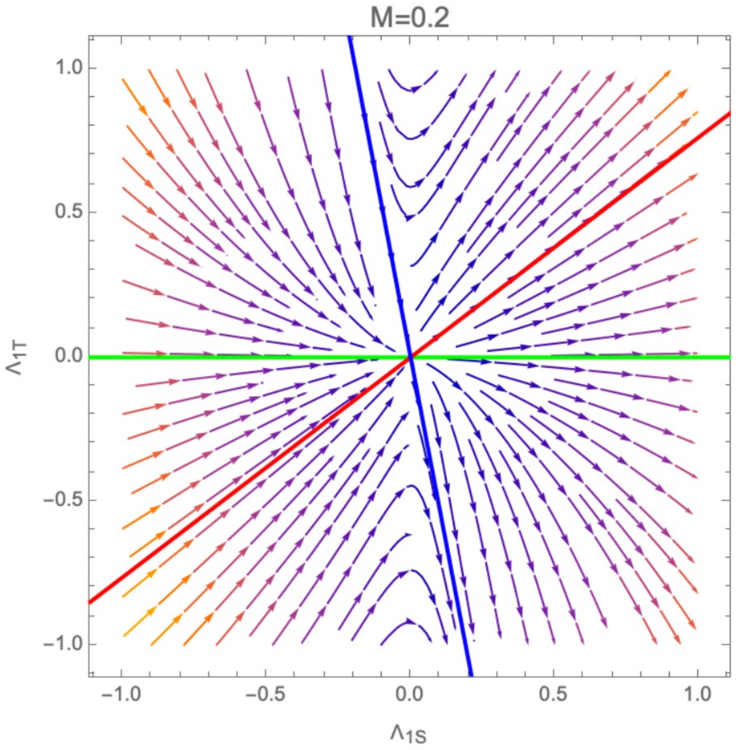
RG flow

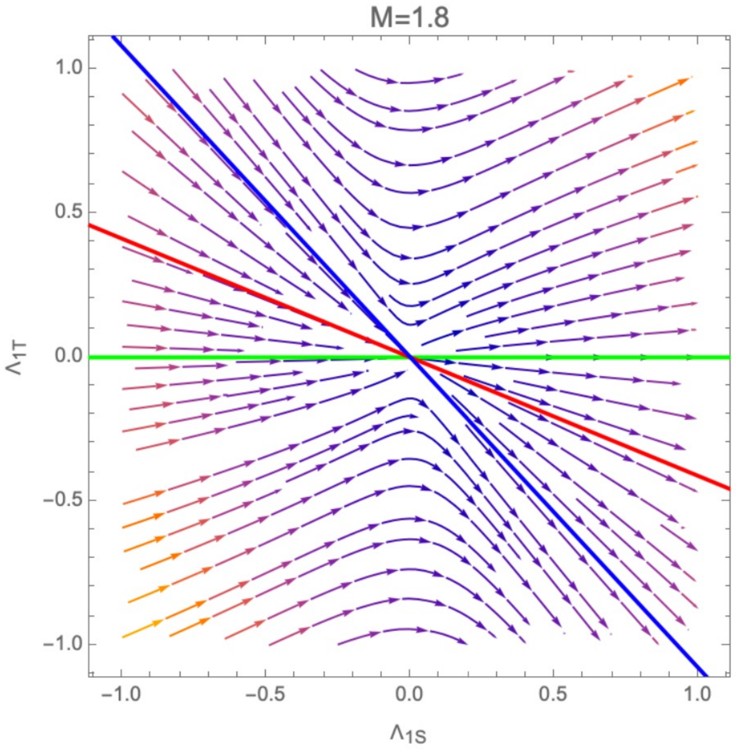
- Case a=0 (M constant)
  - Only trivial fixed point
  - Study 2d simple ideal S^{O2}: 3 idempotents

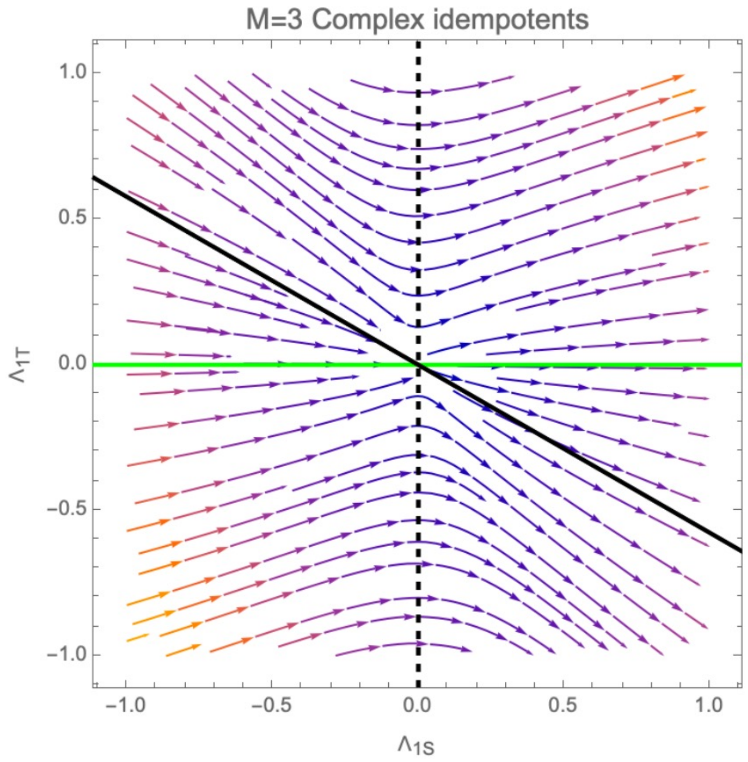
$$
\begin{aligned}
\beta_{1S} =& \frac{1}{32\pi^2 M}((3+M)\lambda_{1S}^2 + 2\sqrt{M}\lambda_{1S}\lambda_{1T} + M\lambda_{1T}^2) \\
\beta_{1T} =& \frac{1}{8\pi^2 M}\lambda_{1S}\lambda_{1T} \\
\beta_{2S} =& \frac{1}{32\pi^2 M}((3+M)\lambda_{1S}^2 + 2\sqrt{M}\lambda_{1S}\lambda_{1T} + 2\lambda_{2S}(2(1+M)\lambda_{1S} \\
& + 2\sqrt{M}\lambda_{1T} + M\lambda_{2S}) + 8M(\lambda_{1S}+\lambda_{2S})\lambda_{2T}) \\
\beta_{2T} =& \frac{1}{128\pi^2 M^2}(4\lambda_{1S}^2 + 8\sqrt{M}\lambda_{1S}\lambda_{1T} + M(2+M)\lambda_{1T}^2 \\
& + 16M\lambda_{2T}(\lambda_{1S} + \sqrt{M}\lambda_{1T} + M\lambda_{2T})).
\end{aligned}
$$

RG flow

- $(\lambda_{1S}, \lambda_{1T})$-space $\quad S^{O2} \cong A/I^2 \quad I^2 = \{e^{2S}, e^{2T}\} \quad S^{O2} = I^2_\perp$
- Idempotents all real for M≤2







Conclusion

- Algebra can be decomposed into a set of simple ideals, each corresponding to a closed subset of couplings with decoupled RG flows
  - Large N: 1d and 2d subspaces of couplings
- The adjoint multiscalar model with SU(N)xO(M) symmetry has 3 large N limits that are easily identified by the algebra
- Positive definite bilinear forms of commutative and non-associative algebras increase the power of algebraic methods

Outlook

- Applying the algebra to other models
  - Algebraic structure for 3-point couplings?
- Higher loops and/or 1/N corrections at large N
  - Does the separation of flows remain?

Thank you!

Decompositions at large N

- Case a=2 (v=M/N²) $A = S^{2S} \oplus (S^{2S}_{\perp} \cap I^S) \oplus S^{OS}$

1d 1d 2d

$S^{OS} \cong A/I^S$

$\diamond$	e^{1T}	e^{2T}
e^{1T}	$\frac{1}{8}e^{2T}$	$2ve^{1T}$
e^{2T}	$2ve^{1T}$	$(2+2v)e^{2T}$

- Case a=0 (M constant) $A = S^{2S} \oplus (S^{2S}_{\perp} \cap I^2) \oplus S^{O2}$

1d 1d 2d

$S^{O2} \cong A/I^2$

$\diamond$	e^{1S}	e^{1T}
e^{1S}	$\frac{M+3}{2M}e^{1S}$	$\frac{1}{2\sqrt{M}}e^{1S} + \frac{1}{M}e^{1T}$
e^{1T}	$\frac{1}{2\sqrt{M}}e^{1S} + \frac{1}{M}e^{1T}$	$\frac{1}{2}e^{1S}$

$I^S = \{e^{1S}, e^{2S}\}$

$S^{OS} = I^S_{\perp}$

$S^{2S} = \{e^{2S}\}$

$I^2 = \{e^{2S}, e^{2T}\}$

$S^{O2} = I^2_{\perp}$