Carrollian fluids and spontaneously broken boosts

Based on 2308.10594 with Jay Armas

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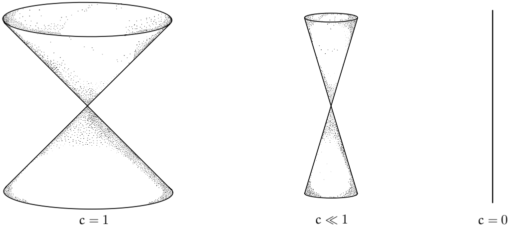
Carrollian symmetry & geometry: redux

- Physically $c \to 0$; hence "ultra-local"



- Generically arises as the geometry on a null hypersurface: in particular black hole horizons & at the infinities of asymptotically flat spacetime [Duval et al., '14; Hartong, '15; Figueroa-O'Farrill et al., '21]

- ...and as the $c \to 0$ limit of Lorentzian geometry [Hansen et al., '21]

Metric $g_{\mu\nu}$ replaced by $(v^{\mu}, h_{\mu\nu})$ satisfying $v^{\mu} h_{\mu\nu} = 0$

What is it good for?

- The main motivation to study Carrollian physics comes from *flat space holography*

  [Bagchi et al., '16; Pasterski et al., '17; Donnay et al., '22]

- Black hole membrane paradigm

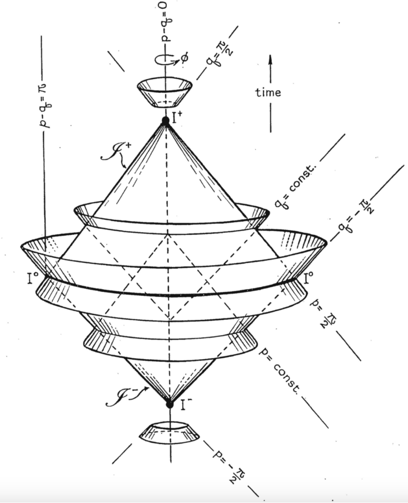
  [Price & Thorne, '86; Penna, '18; Donnay & Marteau., '19]

- Magic angles in superconducting twisted bilayer graphene

  [Bagchi et al., '22]

- Other motivations include the Carroll/fracton duality

  [Figueroa-O'Farrill et al., '23]



Plan

1. Carrollian geometry from expansions
2. $c \to 0$ limit of relativistic fluids
3. Carrollian fluids and the boost Goldstone

Carrollian geometry

A neat way to get a Carrollian geometry is to c^2 expand a Lorentzian geometry: [Hansen et al., '21]

- Write metric and inverse as (*PUL parameterisation*)

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}\,, \quad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}$$

where $T_\mu V^\mu = -1,\ V^\mu \Pi_{\mu\nu} = T_\mu \Pi^{\mu\nu} = 0$

- Expand PUL variables in powers of c^2

$$T_\mu = \tau_\mu + \mathcal{O}(c^2)\,, \qquad V^\mu = v^\mu + \mathcal{O}(c^2)\,,$$
$$\Pi_{\mu\nu} = h_{\mu\nu} + \mathcal{O}(c^2)\,, \qquad \Pi^{\mu\nu} = h^{\mu\nu} + \mathcal{O}(c^2)$$

- Local Lorentz boosts turn into Carrollian boosts with parameter λ_μ (note: $v^\mu \lambda_\mu = 0$)

$$\delta_C \tau_\mu = \lambda_\mu\,, \qquad \delta_C h^{\mu\nu} = 2\lambda_\rho h^{\rho(\mu} v^{\nu)}$$

The connection

- $\hat{\nabla}$ Levi–Civita connection with Christoffel symbols written in terms of the PUL variables as

$$\hat{\Gamma}^{\rho}_{\mu\nu} = -\frac{1}{c^2} V^{\rho} \mathcal{K}_{\mu\nu} + \boxed{\tilde{C}^{\rho}_{\mu\nu}} + \Pi^{\rho\lambda} T_{\nu} \mathcal{K}_{\mu\lambda} + \mathcal{O}(c^2)$$

where $\mathcal{K}_{\mu\nu} = -1/2\, \pounds_V \, \Pi_{\mu\nu}$

- Carrollian adapted "affine" connection $\tilde{\nabla}$ with

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \tilde{C}^{\rho}_{\mu\nu}\Big|_{c=0}, \qquad \tilde{\nabla}_{\mu} v^{\nu} = \tilde{\nabla}_{\mu} h_{\nu\rho} = 0, \qquad \delta_C \tilde{\Gamma}^{\rho}_{\mu\nu} \neq 0$$

NB: Actual affine connection requires the Ehresmann connection ("strong Carrollian geometries") $\leftrightsquigarrow$ fractons?

Revisiting the $c \to 0$ limit of relativistic fluids

Relativistic fluid has U^μ satisfying $U^\mu U^\nu g_{\mu\nu} = -c^2$. In PUL variables:

$$U^\mu = -V^\mu - c^2 \mathfrak{u}^\mu$$

(cf. also [de Boer et al., '23])

for some $\mathfrak{u}^\mu = \theta^\mu + \mathcal{O}(c^2)$.

- Since $\delta V^\mu = c^2 h^{\mu\nu}\lambda_\nu + \mathcal{O}(c^4)$

$$\Rightarrow \boxed{\delta_C \theta^\mu = -h^{\mu\nu}\lambda_\nu}$$

- EMT given by

$$T^\mu{}_\nu = \frac{\hat{\mathcal{E}} + \hat{P}}{c^2} U^\mu U_\nu + \hat{P}\delta^\mu_\nu = (\mathcal{E} + P) v^\mu \hat{\tau}_\nu + P\delta^\mu_\nu + \mathcal{O}(c^2)$$

with

$$\hat{\tau}_\mu = \tau_\mu + h_{\mu\nu}\theta^\nu \,, \qquad \delta_C \hat{\tau}_\mu = 0$$

Broken boosts in Nature

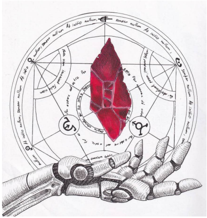
- Thermal states break boost symmetry spontaneously due to preferred rest frame aligned with thermal vector $\beta^\mu = u^\mu / T$ [Alberte+Nicolis, '20; Komargodski et al., '21]

- Relativistic fluids: boost Goldstone normally absorbed in fluid velocity (except for *framids*) [Nicolis et al., '15]

$$\Downarrow$$

Not the case for Carrollian fluids

The Carroll boost Goldstone



The Carroll boost Goldstone transforms as

$$\delta_C \theta^\mu = -h^{\mu\nu}\lambda_\nu$$

NB: Only spatial part of of θ^μ is physical, so we endow with a timelike Stueckelberg symmetry

$$\delta_S \theta^\mu = \chi v^\mu$$

- We can use θ^μ to build the C and S invariant objects

$$\hat{\tau}_\mu = \tau_\mu + h_{\mu\nu}\theta^\nu \, , \quad \hat{h}^{\mu\nu} = h^{\mu\nu} + v^\mu v^\nu(\theta^2 + 2\tau_\rho\theta^\rho) + 2v^{(\mu}\theta^{\nu)}$$

$\Rightarrow$ The fields $(\hat{\tau}_\mu, v^\mu, h_{\mu\nu}, \hat{h}^{\mu\nu})$ form an *Aristotelian* structure

Currents & conservation laws

General variation of free energy $S[\tau_\mu, h_{\mu\nu}; \theta^\mu]$

$$\delta S = \int \mathrm{d}^{d+1}x\, e\left(-\, T^\mu\, \delta\tau_\mu + \frac{1}{2}\, \mathcal{T}^{\mu\nu}\, \delta h_{\mu\nu} - K_\mu\, \delta\theta^\mu \right)$$

- Energy current (T^μ)
- Momentum-stress tensor ($\mathcal{T}^{\mu\nu}$)
- Goldstone equation of motion (K_μ)

The Ward identities for timelike Stueckelberg transformations and Carrollian boosts are

$$v^\mu K_\mu = 0\,, \qquad T^\nu h_{\nu\mu} = K_\mu$$

The diffeo. WI is $\boxed{e^{-1}\partial_\mu(e T^\mu{}_\rho) + T^\mu \partial_\rho \tau_\mu - \frac{1}{2}\mathcal{T}^{\mu\nu}\partial_\rho h_{\mu\nu} = 0}$

where $T^\mu{}_\nu = -\tau_\nu T^\mu + \mathcal{T}^{\mu\rho} h_{\rho\nu}$ (NB: $\delta_{C,S} T^\mu{}_\nu = K_\nu h^{\mu\rho}\lambda_\rho - \chi v^\mu K_\nu$)

Hydrostatic partition function for Carrollian fluids

Given a background Carrollian KVF k^μ, can build gauge inv. scalars

$$T = T_0/\hat{\tau}_\mu k^\mu \,, \qquad \vec{u}^2 = h_{\mu\nu} u^\mu u^\nu$$

where $u^\mu = k^\mu/\hat{\tau}_\rho k^\rho$. NB: *no* temperature unless boosts broken

- Hydrostatic partition function given by

$$S = \int d^{d+1}x \, e \, P(T, \vec{u}^2)$$

$\Rightarrow$ Ideal EMT and Goldstone equation of motion given by

$$T^\mu_{(0)\nu} = P\delta^\mu_\nu + m u^\mu \vec{u}_\nu - (sT + m\vec{u}^2)(u^\mu \hat{\tau}_\nu + \theta^\mu \vec{u}_\nu) \,,$$
$$K_{(0)\mu} = (sT + m\vec{u}^2)\vec{u}_\mu = (\mathcal{E} + P)\vec{u}_\mu$$

- *Two* solutions to $K_{(0)\mu} = 0$: either $\vec{u}_\mu = 0$ or $\mathcal{E} + P = 0$

Summary and outlook

What we have achieved

- The Carroll boost Goldstone plays a crucial rôle in Carrollian fluids
- The $c \to 0$ limit of relativistic fluids gives rise to one branch of the Carrollian fluid
- First order hydro from Aristotelian fluids & dissipative modes

What lies ahead

- Relation to the membrane paradigm?
- "Strong Carrollian fluids" and fractons?
- Relation to flat space holography?

Thank you for your attention

Fluid equations of motion

Relativistic equations of motion

$$\hat{\nabla}_\mu T^\mu{}_\nu = 0$$

turn into

$$\begin{aligned} v^\mu \partial_\mu \mathcal{E} &= (\mathcal{E} + P) K \,, \\ h^{\lambda\mu} \partial_\mu P &= -(\mathcal{E} + P)(h^{\lambda\mu} v^\nu \tau_{\nu\mu} - h^{\lambda\mu} \theta^\nu K_{\mu\nu} - K h^{\lambda\sigma} h_{\sigma\rho} \theta^\rho) \\ &\quad - h^{\lambda\mu} h_{\mu\rho} v^\nu \tilde{\nabla}_\nu ((\mathcal{E} + P)\theta^\rho) \end{aligned}$$

where $\tau_{\mu\nu} = (d\tau)_{\mu\nu}$ and $K_{\mu\nu} = -1/2 \; \pounds_v h_{\mu\nu}$

- If in coordinates $x^\mu = (t, x^i)$ such that Carrollian structure of "Randers–Papapetrou form"

$$v^\mu \partial_\mu = -\frac{1}{\Omega} \partial_t \,, \quad h_{\mu\nu} dx^\mu dx^\nu = a_{ij} dx^i dx^j \,, \quad \tau_\mu dx^\mu = \Omega dt - b_i dx^i$$

recover Carrollian fluids of [Ciambelli et al., '18; Petkou et al., '22; Bagchi et al., '23]

Extra slide: the Carroll algebra

The Poincaré algebra $\mathfrak{iso}(d,1) = \langle P_m, L_{mn} \rangle$ has brackets

$$[L_{mn}, L_{pq}] = \eta_{np} L_{mq} - \eta_{mp} L_{nq} - \eta_{nq} L_{mp} + \eta_{mq} L_{np}$$
$$[L_{mn}, P_p] = \eta_{mp} P_n - \eta_{np} P_m$$

where $\eta_{mn} = (-c^2, 1, \dots, 1)$. Setting $B_a = L_{0a}$ and $H = P_0$, we get

$$[L_{ab}, L_{cd}] = \delta_{ac} L_{bd} - \delta_{bc} L_{ad} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} \qquad [P_a, B_b] = \delta_{ab} H$$
$$[L_{ab}, B_c] = \delta_{ac} B_b - \delta_{bc} B_a \qquad [B_a, B_b] = -c^2 L_{ab}$$
$$[L_{ab}, P_c] = \delta_{ac} P_b - \delta_{bc} P_a \qquad [H, B_a] = c^2 P_a$$

The Carroll algebra is the $c \to 0$ limit of this (with $B_a \to C_a$)

$$[P_a, C_b] = \delta_{ab} H \,, \qquad [L_{ab}, P_c] = \delta_{ac} P_b - \delta_{bc} P_a \,,$$
$$[L_{ab}, C_c] = \delta_{ac} C_b - \delta_{bc} C_a \,,$$
$$[L_{ab}, L_{cd}] = \delta_{ac} L_{bd} - \delta_{bc} L_{ad} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} \,.$$

[Lévy-Leblond, '65; Gupta, '66]