Heterotic Little String Theories (LSTs) – F-theory Geometric Engineering

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What is 6d LST and Why?

- LSTs: **intermediate** between local and gravitational theories, **related by decompactification**

Decoupled from gravity (6D LST → 6D SCFT)

6D SUGRA	→	6D LST	→	6D SCFT
Scale: ✓		**Scale:** ✓		**Scale:** ×
Global Symmetry: ×		**Global Symmetry:** ✓		**Global Symmetry:** ✓
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  1. Contains interesting **global symmetries in the T-dual network**
  2. LSTs have a **2-group structure** $\rightarrow$ Constrain T-dualities
     [Cordova, Dumitrescu, Intriligator'18,'20, Del Zotto, Ohmori'20]
  3. **Systematically engineered** in F-theory geometry
     [Bhardwaj, Del Zotto, Heckman, Morrison, Rudelius, Vafa'15]

Motivation and Goal

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  - **Heterotic strings on ALE spaces** $\mathcal{X}_{\mathfrak{g}} = \mathbb{C}^2/\Gamma_{\mathfrak{g}}$ give rise to **6D (1,0) LSTs** and inherit T-duality
  - LSTs associated to Heterotic $Spin(32)/\mathbb{Z}_2$ ALE instantons are Lagrangian and **well-known**
  - Most of LSTs for Heterotic $E_8 \times E_8$ ALE instantons are **unknown**, few **exceptions** are studied

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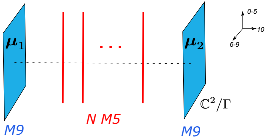
- **Method**: F-theory geometric construction is powerful:
  1. Explore ALE spaces of E-types, **beyond brane realizations**
  2. Easily **interpret 6D LST** (packed into a quiver) by the **geometric configuration**
  3. Realize **T-dual** network as **inequivalent elliptic fibrations** of the same geometry

Generic $E_8 \times E_8$ heterotic instantonic LSTs in HW picture

- The instantonic LSTs in M-theory $\leftarrow$ A stack of N M5 branes [Hořava, Witten'95]:



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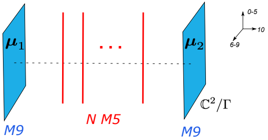

- The resulting theory depends on a choice of a flat connection encoded in:

$$\mu_a : \pi_1(S^3/\Gamma_{\mathfrak{g}}) \simeq \Gamma_{\mathfrak{g}} \to E_8 \,, \quad \text{for } \mu_a \simeq id \,, \text{ see [Aspinwall, Morrison'97]}$$

- The zero form global symmetry is determined by:

$$F_a^{(0)} \equiv \{g \in E_8 \,|\, gh = hg, \forall h \in \mu_a(\Gamma_{\mathfrak{g}})\} \quad a = 1, 2$$

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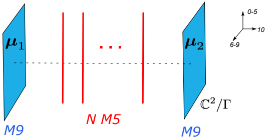

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- Now consider all other **non-trivial** possible choices
  - Determine fractional instantons by **F-theory** [Del Zotto, Heckman, Tomasiello, Vafa'14]

Obtain the full LST

- After the choice of a flat connection encoded in $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$, we denote the corresponding theories:

$$\mathcal{K}_N(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2; \mathfrak{g}) = \mathcal{T}(\boldsymbol{\mu}_1, \mathfrak{g}) \overset{\mathfrak{g}}{\text{———}} \mathcal{T}_{N-2}(\mathfrak{g}, \mathfrak{g}) \overset{\mathfrak{g}}{\text{———}} \mathcal{T}(\boldsymbol{\mu}_2, \mathfrak{g})$$

  - $\mathcal{T}(\boldsymbol{\mu}_a, \mathfrak{g})$: minimal 6d orbi-instanton theory associated to a single M9-M5 system
    [Heckman, Morrison, Rudelius, Vafa'15, Mekareeya, Ohmori, Tachikawa, Zafrir'17, Frey, Rudelius'18]
  - $\mathcal{T}_{N-2}(\mathfrak{g}, \mathfrak{g})$: 6d conformal matter theory associated to $N-2$ M5 branes
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  - $\overset{\mathfrak{g}}{\text{———}}$: fusion of the common factors $\mathfrak{g}$ of the global symmetry of SCFTs
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- Given the matching criteria (see next slide), we are able to **chart the T-duality**:

$$\mathcal{K}_N(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2; \mathfrak{g}) \sim \widetilde{\mathcal{K}}_{\widetilde{N}}(\boldsymbol{\lambda}; \widetilde{\mathfrak{g}}) \text{ (Heterotic } Spin(32)/\mathbb{Z}_2 \text{ instatons)}$$

6d LSTs of rank n_T ↔ Quiver notations encoded by

$$[\mathfrak{f}_1] \overset{\mathfrak{g}_1}{n_1} \cdots \overset{\mathfrak{g}_I}{n_I} \cdots \overset{\mathfrak{g}_{n_T+1}}{n_{n_T+1}} [\mathfrak{f}_{n_f}]: \qquad \begin{pmatrix} \eta^{IJ} & \eta^{IA} \\ \eta^{AI} & 0 \end{pmatrix} \qquad \begin{matrix} I, J = 1, \ldots, n_T + 1 \\ A = 1, \ldots, n_f \end{matrix}, \qquad \mathfrak{g} = (\mathfrak{g}_1, \ldots, \mathfrak{g}_{n_T+1}, \mathfrak{f}_1, \ldots, \mathfrak{f}_{n_f})$$

- The Dirac pairing η^{IJ}: **non-negative** with a **unique null eigenvector**:

$$\eta^{IJ} N_J = 0 \qquad \gcd(N_1, \ldots, N_{n_T+1}) = 1 \qquad N_I > 0 \,(\text{LST charges})$$

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LSTs have generic continuous 2-group symmetry with matching criteria

$$\left(\mathscr{P}^{(0)} \times SU(2)_R^{(0)} \times \textstyle\prod_a F_a^{(0)}\right) \times_{\widehat{\kappa}_{\mathscr{P}},\widehat{\kappa}_R,\widehat{\kappa}_{F_a}} U(1)_{LST}^{(1)}$$

- $\widehat{\kappa}_{\mathscr{P}}$, $\widehat{\kappa}_R$ and $\widehat{\kappa}_{F_a}$ (2-group structure constants)

$$\widehat{\kappa}_F = -\sum_{I=1}^{n_T+1} N_I \eta^{IA} \qquad \widehat{\kappa}_R = \sum_{I=1}^{n_T+1} N_I h^\vee_{\mathfrak{g}_I} \qquad \widehat{\kappa}_{\mathscr{P}} = -\sum_{I=1}^{n_T+1} N_I (\eta^{II} - 2)$$

- Coulomb branch dimension and amounts of Wilson line parameters

$$\mathbf{Dim(CB)} = T + \text{rk}(G)\,, \qquad \mathbf{Dim(WL)} = \text{rk}(G_F)$$

F-theory in a nutshell [Vafa'96, 2017 TASI lectures by Weigand and Cvetič, 2018 6D SCFT review by Heckman and Rudelius.....]

- Axio-dilation field τ in IIB $\leftrightarrow$ the behaviour of the complex structure of an elliptic curve
- Introduce an auxillary torus to record $\tau \leftrightarrow$ An **elliptic fibration** $\pi: Y_n \twoheadrightarrow B_{n-1}$

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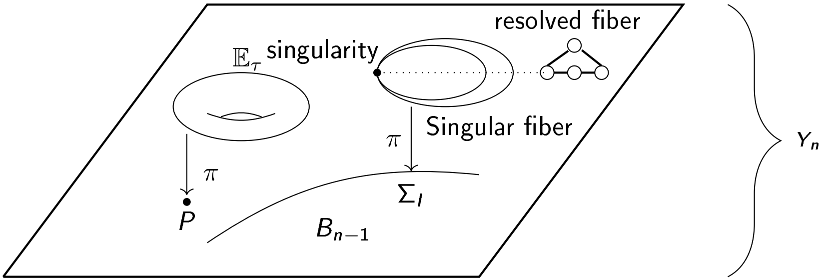
  1. Elliptic fibre becomes **singular** over **discriminant** locus $\Sigma \subset B_{n-1}$
  2. **D7 brane** stacks are **located at component** Σ_I of the discriminant
  3. After resolution, the fiber components over Σ_I has intersection pattern as affine Dynkin diagram $\rightarrow$ **Fibration** consistently **realize gauge/flavor algebra**



Review: Geometric Engineer 6D LST from F-theory

- An elliptic fibered CY threefold X $\xleftrightarrow[\text{Geometric Engineering}]{\text{Het/F Duality}}$ Physics of a 6d Heteorotic LST

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  - B_2 is **non-compact** $\leftrightarrow$ LST is **decoupled from gravity**
  - The **intersection form** of the F-theory **base curve collection** $\leftrightarrow$ **Dirac pairing matrix**
  - Discriminant admits compact and non-compact components that D7 branes can wrap:
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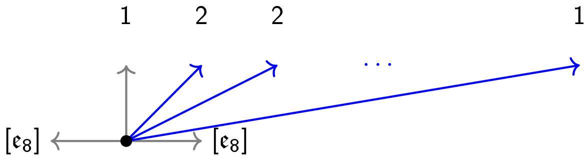
- X may admit **multiple inequivalent** elliptic fibrations:

$$\begin{array}{ccc} T^2 \to & X & \\ & \downarrow \pi & \\ & B_2 & \end{array} , \quad \begin{array}{ccc} T^2 \to & X & \\ & \downarrow \widetilde{\pi} & \\ & \widetilde{B_2} & \end{array}$$

  - After circle reduction, obtain the same 5d theory (inequivalent 6d uplifts)
  - **Geometrically realize T-duality** between these two 6d theories

F-theory Realization of 6D LSTs - Warmup

1. An elliptic fibered CY_3 with generic base $B_2 = \mathbb{P}^1 \times \mathbb{C}$ supports $E_8 \times E_8$ flavor algebra
2. Left: decorate additional rays, yield N small E_8 instantons on the $E_8 \times E_8$ fibration



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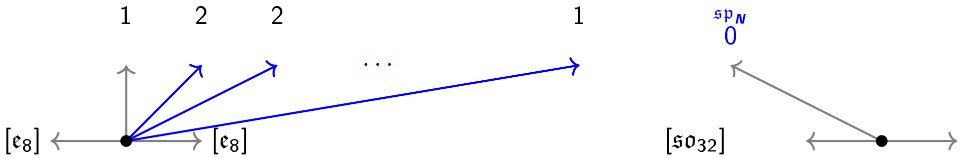

3. Right: T-dual $Spin(32)/\mathbb{Z}_2$ theory with $\mathfrak{sp}_N$ gauge group [Aspinwall, Morrison'97]
4. Consistently check for T-dual pairs by the matched data:

$$\mathrm{Dim(CB)} = N + 1\ , \qquad \widehat{\kappa}_{\mathscr{R}} = h^{\vee}_{\mathfrak{g}} = N + 1\ , \qquad \widehat{\kappa}_{\mathscr{P}} = 2$$

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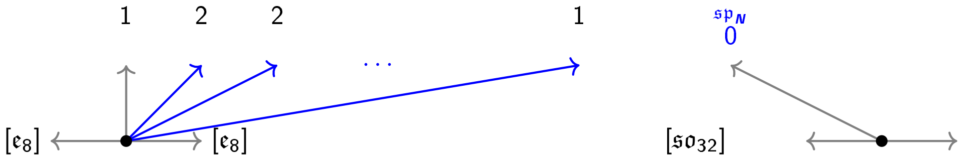

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5. Additional rays can support **non-trivial** gauge algebra in more general constructions $\leftrightarrow$ T-dual configuration will be altered accordingly

A general example: The $[\mathfrak{e}_8] - \mathfrak{e}_7^M - [\mathfrak{e}_6]$ LST

- Gauging the $E_8 \times E_6$ flavor factors with $M\times$ $\mathfrak{e}_7$'s, we obtain

$$[\mathfrak{e}_8] \; \underbrace{\overset{\mathfrak{e}_7}{1} \; \overset{\mathfrak{e}_7}{2} \dots \overset{\mathfrak{e}_7}{2} \dots \overset{\mathfrak{e}_7}{2} \; \overset{\mathfrak{e}_7}{1}}_{\times M} \; [\mathfrak{e}_6]\,, \text{ M-1 also counts number of M5 branes}$$

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- The T-dual fibration is given by:

$$[\mathfrak{u}_1 \times \mathfrak{so}_{26}] \; \underset{[N_F=1]}{\overset{\mathfrak{sp}_{M+7}}{1}} \; \overset{\mathfrak{so}_{4M+16}}{4} \; \overset{\mathfrak{sp}_{3M+1}}{1} \; \overset{\overset{\mathfrak{sp}_{2M-3}}{1^*}}{\overset{\mathfrak{so}_{8M+4}}{4^*}} \; \overset{\mathfrak{sp}_{3M-2}}{1} \; \underset{[N_F=1]}{\overset{\overset{\mathfrak{sp}_{M-3}}{1^*}}{\overset{\mathfrak{so}_{4M+4}}{4^*}}}\,,$$

They have matched data: $\text{Dim(CB)} = 18M + 19\,, \quad \widehat{\kappa}_{\mathscr{R}} = 48M + 25$

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- **Fiber-Base-Duality:** Most T-dual base topology are determined by gauging of the original theory

LSTs with Discrete Holonomy

- Consider a **non-trivial global structure** [Aspinwall, Morrison'98...]

$$MW(X) = \mathbb{Z}^r \times MW(X)_{Tor} \Rightarrow G_T = \frac{G_F \times G}{MW_{Tor}}$$

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$$[\mathfrak{e}_7]\ \underbrace{\overset{\mathfrak{so}_{4N+8}}{\underset{[\mathfrak{su}_2]}{1}}\ \overset{\mathfrak{so}_{4N+8}}{2}\ \ldots\ \overset{\mathfrak{so}_{4N+8}}{2}\ \overset{\mathfrak{so}_{4N+8}}{\underset{[\mathfrak{su}_2]}{1}}}_{M\times}\ [\mathfrak{e}_7]$$

- Has two more inequivalent toric fibrations, first:

$$\begin{array}{cccccccccc} & & \overset{\mathfrak{sp}_{2N+M-3}}{1} & & & & & & \overset{\mathfrak{sp}_{M-3}}{1^*} & \\ [\mathfrak{so}_{24}] & \overset{\mathfrak{sp}_{2N+M+3}}{1} & \overset{\mathfrak{so}_{8N+4M+4}}{4} & \underbrace{\ldots\ \overset{\mathfrak{sp}_{4(N-k)+2M-4}}{1}\ \overset{\mathfrak{so}_{8(N-k)+4M-4}}{4}\ \ldots\ \overset{\mathfrak{sp}_{2M}}{1}\ \overset{\mathfrak{so}_{4M+4}}{4^*}}_{2N\times} & & & & & \overset{\mathfrak{sp}_{M-1}}{1} & [\mathfrak{so}_8] \end{array}$$

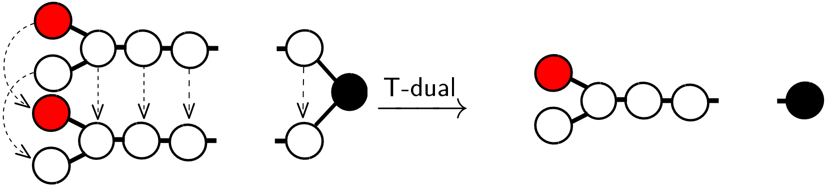
This chain has the full $\mathfrak{so}_{4N+8}^{(1)}$ topology! $\leftrightarrow$ **Fiber-Base-Duality**

$\mathbb{Z}_2$ Discrete Holonomy LSTs

- The third fibration has the quiver:

$$\begin{array}{cc} [\mathfrak{su}_{16}\times\mathfrak{u}_1] & \overset{\mathfrak{su}_{4N+2M+6}}{2} \\ & \overset{\mathfrak{su}_{8N+4M-4}}{2} \quad \underbrace{\overset{\mathfrak{su}_{8N+4M-12}}{2}\ \ \overset{\mathfrak{su}_{8N+4M-20}}{2}\ \ \ldots\ \ \overset{\mathfrak{su}_{4M+4}}{2}\ \ \overset{\mathfrak{sp}_{2M-2}}{1}}_{N\times} \\ & \overset{\mathfrak{su}_{4N+2M-2}}{2} \end{array}$$

- $\mathfrak{so}^{(1)}_{4N+8}$ base shape is folded to an $\mathfrak{su}^{(2)}_{N+3}$



- The 2-groups and CB dimension are matched and given below as

$$\mathrm{Dim(CB)} = 4N^2 + 4NM + 8N + 6M + 2\,, \quad \widehat{\kappa}_{\mathscr{R}} = 8N^2 + 8NM + 8N + 8M + 2$$

Geometric engineering of heterotic strings

- No full M5 branes but only the M9 fractions:

  1. Orbi-instanton quiver → Reduced theories → Fuse two reduced theories:

$$\mathcal{K}_0(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2; \mathfrak{g}) = \mathcal{T}_{red}(\boldsymbol{\mu}_1; \mathfrak{g}) \overset{\mathfrak{g}}{\text{———}} \widehat{\mathcal{T}}_{red}(\boldsymbol{\mu}_2; \mathfrak{g})$$

$$= [\mathfrak{f}(\boldsymbol{\mu}_1)] \overset{\mathfrak{h}_1}{n_1} \overset{\mathfrak{h}_2}{n_2} \ldots \overset{\mathfrak{h}_{r_1}}{n_{r_1}} \overset{\mathfrak{g}}{m_{r_1+1}} \overset{\widehat{\mathfrak{h}}_{r_2}}{\widehat{n}_{r_2}} \ldots \overset{\widehat{\mathfrak{h}}_2}{\widehat{n}_2} \overset{\widehat{\mathfrak{h}}_1}{\widehat{n}_1} [\mathfrak{f}(\boldsymbol{\mu}_2)]$$

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2. For A-type singuarities:

$$\mathcal{K}_0(E_8,E_8;\mathfrak{su}_k) = [E_8]\ 1\ 2\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{su}_3}{2}\ \cdots\ \overset{\mathfrak{su}_{k-1}}{2}\ \underset{N_f=2}{\overset{\mathfrak{su}_k}{2}}\ \overset{\mathfrak{su}_{k-1}}{2}\ \cdots\ \overset{\mathfrak{su}_3}{2}\ \overset{\mathfrak{su}_2}{2}\ 2\ 1\ [E_8]$$

3. If k is even, the T-dual theory is

$$\widetilde{\mathcal{K}}(\mathfrak{so}_{32};\mathfrak{su}_k) = [\mathfrak{so}_{32}]\ \overset{\mathfrak{sp}_{2k}}{1}\ \overset{\mathfrak{su}_{4k-8}}{2}\ \overset{\mathfrak{su}_{4k-16}}{2}\ \cdots\ \overset{\mathfrak{su}_8}{2}\ 1\ [\mathfrak{su}_2]$$

4. If k is odd, the T-dual theory is

$$\widetilde{\mathcal{K}}(\mathfrak{so}_{32};\mathfrak{su}_k) = [\mathfrak{so}_{32}]\ \overset{\mathfrak{sp}_{2k}}{1}\ \overset{\mathfrak{su}_{4k-8}}{2}\ \overset{\mathfrak{su}_{4k-16}}{2}\ \cdots\ \overset{\mathfrak{su}_{12}}{2}\ \underset{N_A=1}{\overset{\mathfrak{su}_4}{1}}$$

Fronzen Singularity and Incomplete fusion

- In M-theory, can partially freeze the $\mathcal{X}_{\mathfrak{g}}$ singularities by torsional C_3 fluxes at infinity

  [De Boer, Dijkgraaf, Hori, Keurentjes, Morgan, Morrison, Sethi'01, Tachikawa'15]

$$\int_{S^3/\Gamma_{\mathfrak{g}}} C = \frac{\ell}{d}$$

- This results subalgbra

$\mathfrak{g}$	$\mathfrak{so}_{2k+8}$	$\mathfrak{e}_6$		$\mathfrak{e}_7$			$\mathfrak{e}_8$				
$\frac{l}{d}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3},\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3},\frac{2}{3}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{3},\frac{2}{3}$	$\frac{1}{4},\frac{3}{4}$	$\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$	$\frac{1}{6},\frac{5}{6}$
$\mathfrak{h}$	$\mathfrak{sp}_k$	$\mathfrak{su}_3$	—	$\mathfrak{so}_7$	$\mathfrak{su}_2$	—	$\mathfrak{f}_4$	$\mathfrak{g}_2$	$\mathfrak{su}_2$	—	—

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$\mathfrak{h}$	$\mathfrak{sp}_k$	$\mathfrak{su}_3$	—	$\mathfrak{so}_7$	$\mathfrak{su}_2$	—	$\mathfrak{f}_4$	$\mathfrak{g}_2$	$\mathfrak{su}_2$	—	—

- $\rightarrow$ An incomplete fusion (decouple the quiver at the $i-$th node) **[Mekareeya, Ohmori, Shimizu, Tomasiello'17]**

$$\mathcal{T}_{red}(\mu;\mathfrak{g}) = [\mathfrak{f}(\mu)] \overset{\mathfrak{h}_1}{n_1} \overset{\mathfrak{h}_2}{n_2} \ldots \overset{\mathfrak{h}_i}{n_i} \ldots \overset{\mathfrak{h}_r}{n_r} [\mathfrak{g}]$$

Exceptional non-simply laced fusion

- Consider $\mathcal{T}_{red}(\mathfrak{e}_6 : \mathfrak{e}_8) \rightarrow$ Read off the unbroken gauge algebra: e.g $\mathfrak{f}_4$

$$[\mathfrak{e}_6]\,1\,\overset{\mathfrak{su}_3}{2}\,1\,\underbrace{\overset{\mathfrak{f}_4}{5}\,1\,\overset{\mathfrak{g}_2}{3}\,\overset{\mathfrak{sp}_2}{2}\,2\,1\,[\mathfrak{e}_8] \quad \text{and} \quad [\mathfrak{e}_8]\,1\,2\,\overset{\mathfrak{sp}_1}{2}\,\overset{\mathfrak{g}_2}{3}\,1\,\overset{\mathfrak{f}_4}{5}}_{\text{Fusion}}\,1\,\overset{\mathfrak{su}_3}{2}\,1\,[\mathfrak{e}_6]$$

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- Yield LST :

$$[\mathfrak{e}_6]\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\underset{[N_F=1]}{\overset{\mathfrak{f}_4}{4}}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,[\mathfrak{e}_6]$$

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- This LST has a T-dual partner

$$[\mathfrak{so}_{20}]\,\overset{\mathfrak{sp}_4}{1}\,\overset{\mathfrak{so}_{12}}{4}\,\underset{[\mathfrak{su}_2^2]}{1}\,2\,2\,[\mathfrak{su}_2]$$

Summary and Outlook

- **Summary:**
  1. Construct a plethora of T-dual LSTs probing various types of singularity verified by Coulomb and tensor branch data matching
  2. Find interesting exotic LSTs with torsional structure
  3. Explore geometric engineering limit of Heterotic strings

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- **Outlook:**
  1. Turn towards non-heterotic LSTs given by systems without M9 branes
  2. Incorporate the possibility of twisted compactifications
  3. Relate heterotic LSTs to the underlied nested K3 fibration of CY_3, study unexplored reducible K3 fibrations occur in the geometry of LSTs

Thank you!