Integrable Chains , Gromov – Witten Theory And Supersymmetric Gauge Theories

Wei Gu

Max Planck Institute for Mathematik

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Motivation

In this talk, I will tell you that the integrable system (Chain) can emerge from a two-dimensional supersymmetric gauge theory.
But why do we need this fact?

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- Mathematicians used the techniques of integrable systems to prove the theorems of the Gromov-Witten theory.
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- Physicists found several similarities between the supersymmetric gauge theory and the integrable system. For example, vacuum equations of gauge theory are the same as the Bethe-ansatz equations of the corresponding integrable system.
- Possible others ...

One of my goals in this talk is to convince you that the third point is a fact, then it explains why the three "different subjects" are inherently related.

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Many of you may be already familiar with the integrable chain. It has rich important physics, but the setup is simple.
It has N-sites, each site has a vector space $\mathcal{V}$ and the associated Hamiltonian as

$$H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} \right) + \dots$$

Figure: Spin chain.

Spin Chain

Examples. Consider a spin-1/2 system: $\mathcal{V} = \{+, -\}$.
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The k spin-up configuration corresponds to the configuration of k-magnons.
We will see that these can be mapped to the supersymmetric gauge theory. Magnons map to the vacuum states of supersymmetric gauge theory, and the interaction $S_{+,i} S_{-,i-1}$ and $S_{-,i} S_{+,i-1}$ map to the fundamental domainwalls $\cdots$

Bosonic chain

Compare to spin chain, we can also have the bosonic chain: multiple particles can locate on the same site.

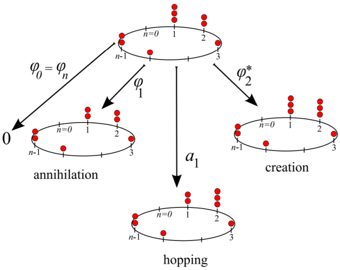

Figure: Bosonic chain.

I will only focus on the q-deformed (-phase) model in this talk, and I will propose a new wave function (Bethe ansatz) guided by symmetries of quantum field theory.

Gr(k;N)

A brief discussion of GLSM with a concrete example.

Gr(k;N)

A brief discussion of GLSM with a concrete example.Gauged linear sigma model for Gr(k;N) is well-studied. See Witten '93, Gu-Sharpe '18 (Exact results in the holomorphic scheme). It is a $U(k)$ gauge theory with N fundamental fields.

- When $r \gg 0$, the semi-classical vacuum configuration is the geometry: Gr(k;N).
- When $r \ll 0$, the vacuum structure is described by a twisted effective superpotential

$$\widetilde{W}_{eff} = -(t + i(k-1)\pi) \sum_{a=1}^{k} \Sigma_a - \sum_a N\Sigma_a \left(\log\left(\Sigma_a\right) - 1\right).$$

  The vacuum equations

$$e^{\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_a}} = 1$$

  give

$$(\sigma_a)^N = \widetilde{q}, \quad \widetilde{q} = e^{-t - i\pi(k-1)}, \qquad \mathrm{a} \in \{1, \cdots, k\}, \quad \sigma_a \neq \sigma_b.$$

We will see the above equations are also the Bethe-ansatz equations of the N-sites XX-spin chain.

BPS Spectrum

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For projective space, or a $U(1)$ gauge theory, it was observed first by Witten '79 that the fundamental domainwall, which connects two adjacent vacua, has a gauge charge 1. A candidate of this domainwall is the IR heavy matter ϕ (In UV, it is a "fundamental field".). The mass of this domainwall is proportional to the dynamical scale $\Lambda^N = q$:

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$$Z \approx \Lambda$$

It has a finite mass, so, unlike the bosonic theory, 2d $\mathcal{N} = (2,2)$ massive theory is not confined. More generally, we have

$$Z_{\ell_1 \ell_2} = W\,|_{\ell_2} - W\,|_{\ell_1}, \qquad \ell \text{ labels the vacua.}$$

See Hannay-Hori '97 and Hori-Vafa '20 for abelian theories. See also Dorey '98 about the connection to the BPS-spectrum in Seiberg-Witten theory. For nonabelian theories, see Gu '22. If turning on the twisted mass, the central charge expression will be modified.

An intermediate scale

In quantum field theory, the physics depends on scale. Now if we consider a scale

$$\Lambda \ll e(\mu),$$

where e is the gauge coupling. The mass of the perturbative spectrum is proportional to the gauge coupling e. At this scale, the physics degrees of freedom are domainwalls. We will show that this is the theory for spin chains.

State Space

Following Witten '82 or Hori.et.al mirror symmetry book '2000, after some operations, we can represent the state-space by a formal notation (Gu '22), for example, for k=2

$$|-, \cdots, +, -, \cdots, -, +, -, \cdots, -\rangle.$$

This notation is not necessary for quantum field theory, however, it would be useful for presenting the data of the spin chain.

Spin Operators

One can construct local spin operators on each site as

$$S_{+,i} = \bar{\lambda}_{+,i}\lambda_{-,i}, \quad S_{-,i} = \bar{\lambda}_{-,i}\lambda_{+,i}, \quad S_{z,i} = \frac{1}{2}\left(\bar{\lambda}_{+,i}\lambda_{+,i} - \bar{\lambda}_{-,i}\lambda_{-,i}\right).$$

It is readily to show that

$$[S_+, S_-] = 2S_z.$$

This explains why we need $\mathcal{N} = (2,2)$ supersymmetry for getting a spin chain. For more general global symmetry in gauge theory, it would have one more index for defining the local spin operators, see Gu '22.

Domain walls In The Spin Chain

The domain wall can be regarded as a map such as

$$|-, \cdots, -_{i-1}, +_i, -_{i+1} \cdots\rangle \mapsto |-, \cdots, -_{i-2}, +_{i-1}, -_i \cdots\rangle$$

$$\phi :\mapsto \mathcal{D}_i = S_{-,i} S_{+,i-1}.$$

One can similarly define the anti-domain wall. For more general domain walls, a similar construction applies.

Constraints From Domain walls

In our case, we have the global symmetry $SU(N)$ in our target space. The domain walls are charged under the center group $\mathbb{Z}_N$ of the flavor group $SU(N)$. So the dynamic domain walls suggested that all of the physical state space should be neutral under the center group $\mathbb{Z}_N$. Notice that, the expectation value of σ field is also charged under $\mathbb{Z}_N$ group (also they are charged under the axial-R symmetry as well), which can be used to construct the states.

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For example, consider the k=1 case, the physical state space at the intermediate scale can be represented as

$$\sum_{i=1}^{N} \sigma^i \mid \cdots +_i \cdots \rangle.$$

This is actually the state for one "magnon" in the spin chain if we replace σ with e^{ip}, where p is the momentum of the magnon.

Finite Symmetries: CPT

CPT-symmetries in quantum field theory can also descend to the spin chain. They act as the following:

$$
\begin{aligned}
\mathcal{P} S_{\pm,i} \mathcal{P}^{-1} &= S_{\mp,i} & \qquad \mathcal{P} S_{z,i} \mathcal{P}^{-1} &= -S_{z,i}, \\
\mathcal{T} S_{\pm,i} \mathcal{T}^{-1} &= -S_{\pm,i} & \qquad \mathcal{T} S_{z,i} \mathcal{T}^{-1} &= S_{z,i}, \\
\mathcal{C} S_{\pm,i} \mathcal{C}^{-1} &= -S_{\mp,i} & \qquad \mathcal{C} S_{z,i} \mathcal{C}^{-1} &= -S_{z,i}.
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These symmetries act on the domain walls as

$$\mathcal{P}\mathcal{D}_i\mathcal{P}^{-1} = \bar{\mathcal{D}}_i \qquad \mathcal{P}\mathcal{D}_N\mathcal{P}^{-1} = \left(\frac{\widetilde{q}}{|\widetilde{q}|}\right)^2 \bar{\mathcal{D}}_N,$$
$$\mathcal{T}\mathcal{D}_i\mathcal{T}^{-1} = \mathcal{D}_i \qquad \mathcal{T}\mathcal{D}_N\mathcal{T}^{-1} = \left(\frac{\widetilde{q}}{|\widetilde{q}|}\right)^{-2} \mathcal{D}_N,$$
$$\mathcal{C}\mathcal{D}_i\mathcal{C}^{-1} = \bar{\mathcal{D}}_i \qquad \mathcal{C}\mathcal{D}_N\mathcal{C}^{-1} = \bar{\mathcal{D}}_N,$$

where we have used the fact that $\mathcal{T}i\mathcal{T}^{-1} = -i$ for $i^2 + 1 = 0$. From the above, one can observe that $\mathcal{P}$ and $\mathcal{T}$ may be violated individually unless $\widetilde{q} = \pm 1$. However, we will see that the scattering factor could also break the $\mathcal{T}$-symmetry if it is not a pure phase factor.

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• When the scattering factor is generic not a pure phase: It is always an open spin chain for any finite $\widetilde{q}$.

Hamiltonian

For a closed spin chain, it is natural to propose the Hamiltonian as

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i + f(\mathcal{D}_i, \bar{\mathcal{D}}_i),$$

where the factor $f(\mathcal{D}_i, \bar{\mathcal{D}}_i)$ can be fixed if we require that the state spaces are eigenstates of this Hamiltonian. We have assumed the fact that all other matter representations are a subspace of the tensor product of fundamental matters. This is not true for general, and for a general case, one may need to take into account the so-called higher symmetries. We will not focus on this in this talk.

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For an open spin chain, we take the (anti-)holomorphic part of the above.

A Summary

Let us summarize what we have. Before, Nekrasov and Shatashvili found about the correspondence between the integrable system and the gauge theory:

$\mathrm{Y}(p_a)$	$\leftrightarrow$	$\widetilde{W}_{\mathrm{eff}}(\sigma_a)$
p_a	$\leftrightarrow$	σ_a
k-particle sector	$\leftrightarrow$	gauge group $U(k)$
N-sites	$\leftrightarrow$	flavor group $SU(N)$
twisted boundary	$\leftrightarrow$	$t = r - i\theta$
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Based on the vacuum structure and the dynamics of the domainwalls. We have shown that the spin chain can emerge from the $\mathcal{N} = (2,2)$ supersymmetric gauge theory in two steps.

- Write the ground states as the Hilbert space formula, and (isomorphic-)map them to the magnon's configuration in the spin chain.
- Kinetic and dynamics of magnons map to the kinetic and dynamics of domainwalls.

So the above similarities found by Nekrasov and Shatashvili between the integrable systems and the gauge theories are just a consequence of our framework.

Examples: Gr(k;N)

Now we focus on the simplest example: gauged linear sigma model for Gr(k;N). The vacuum equations are

$$(\sigma_a)^N = \Lambda^N.$$

If we use the variable $\widetilde{\sigma}_a = \sigma_a \Lambda^{-1}$, the vacuum equations reduce to

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These equations are Bethe-equations of the XX model. And we expect that the spin chain is a closed one.
The Hamiltonian is simple:

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i.$$

The eigenstates are k-magnons

$$\mid k\rangle := A_k \sum_{1 \le j_1 < \cdots < j_k \le N} \prod_{a=1}^{k} \sum_{\mathcal{W} \in S_k} \left(\widehat{\sigma}_{\mathcal{W}(a)}\right)^{j_a} |j_1, \cdots, j_k\rangle_k.$$

The eigenvalue is

$$H \cdot \mid k\rangle := (e_1(\sigma) + e_1(\bar{\sigma})) \mid k\rangle$$

One might have noticed that the holomorphic part of the eigenvalue $e_1(\sigma)$ is the cohomology generator of $H^{1,1}(Gr(k;N))$. One may expect that all of the generators can be constructed as the eigenvalues of the corresponding operators. It was first done by mathematicians (Postnikov '05, C. Korff and C. Stroppel '09) that one can construct a general elementary function:

$$e_r(\mathcal{D}).$$

The eigenvalue of this is $e_r(\sigma)$. One can further construct the Schur operators $S_\lambda\left(\mathcal{D}\right)$ that the eigenvalues are

$$S_\lambda\left(\sigma\right).$$

In this context, the quantum cohomology of Gr(k;N) can be reconstructed from the XX model.

Open XX model

If we start from the vacuum equations

$$(\sigma_a)^N = \Lambda^N.$$

The emergent spin chain is an open one.
All the discussions are similar to the closed one by only focusing on the (anti-)holomorphic part.

K-theoretic of XX model

One can lift gauged linear sigma models to the 3d CS-matter theories on space-time $\mathbb{R}^2 \times S^1$. Compared to the 2d theory, one has one more parameter to label the vacua and others: the so-called Chern-Simons levels. We only focus on the gauge Chern-Simons level $k_{U(k)}$. We split it into two factors: $k_{U(1)}$ and $k_{SU(k)}$.

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- If we choose (Jockers et.al, Ueda and Yoshida '2019, Gu et.al '20 and '22)

$$k_{U(1)} = -\frac{N}{2}, \qquad k_{SU(k)} = k - \frac{N}{2},$$

the vacuum equations are

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the vacuum equations are

$$(1 - x_a)^N = \widetilde{q}\frac{(x_a)^k}{\prod_{b=1}^{k} x_b}.$$

There is "interaction factor" on the right hand side

$$S_{ab} = -\frac{x_a}{x_b}.$$

This factor means the magnons interacts each other. Since this factor of this case is not a pure phase, the spin chain of this case is always an open one regardless of what the value of $\widetilde{q}$ is. This is because if we regard

$$1 - x_a := e^{ip_a}$$

The momentum p_a is not real.

State Space

One can propose the interacting state space as follows:

$$A_k \sum_{1 \leq j_1 < \cdots < j_k \leq N} \sum_{\mathcal{W} \in S_k} S_{\mathcal{W}(a)} e^{\sum_{a=1}^{k} i p_{\mathcal{W}(a)} j_a} |j_1 < \cdots < j_k\rangle_k,$$

where A_k is an overall normalization factor, and $\sum_{\mathcal{W} \in S_k} S_{\mathcal{W}}$ is a multiplication of scattering factors S_{ab}. One can show that the proposed state space is consistent with the symmetries.

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where A_k is an overall normalization factor, and $\sum_{\mathcal{W}\in S_k} S_{\mathcal{W}}$ is a multiplication of scattering factors S_{ab}. One can show that the proposed state space is consistent with the symmetries. The fundamental Hamiltonian of this system is a complex one, which was first proposed in a slightly different context by Gorbounov and Korff '2014:

$$h = \sum_{i}^{N} \mathcal{D}_i - \sum_{|i_1-i_2| \mod N>1}^{N} \mathcal{D}_{i_1}\mathcal{D}_{i_2} + \sum_{|i_a-i_b| \mod N>1}^{N} \mathcal{D}_{i_1}\mathcal{D}_{i_2}\mathcal{D}_{i_3} + \cdots .$$

The number of sites is N, so only finitely many terms act non-trivially, and the series therefore terminates.

Gr(2;N)

Let us check whether the ground state wave functions proposed in our paper for this case are indeed the eigenstates of the Hamiltonian. To shorten the notation, we denote the ground state as $|\,\omega\rangle_2 = \sum_{1\le j_1<j_2\le N} a(j_1,j_2)|j_1 < j_2\rangle_2$, where

$$a(j_1,j_2) = A\left(e^{i(p_1j_1+p_2j_2)} + S_{21}e^{i(p_2j_1+p_1j_2)}\right).$$

The boundary condition is $a(j_1,j_2+N) = q\cdot a(j_2,j_1)$, which gives

$$e^{i(p_1j_1+p_2j_2)}e^{ip_2N} + S_{21}e^{ip_1N}e^{i(p_2j_1+p_1j_2)} = q\left(e^{i(p_1j_2+p_2j_1)} + S_{21}e^{i(p_2j_2+p_1j_1)}\right).$$

The vacuum equations can be read from the above as

$$e^{ip_1N} = qS_{12}, \qquad e^{ip_2N} = qS_{21}.$$

So it is consistent.

Gr(2;N)

To compute $H \mid \omega\rangle_2$, special care is needed when two overturned spins are sitting next to each other. We find

$$\begin{aligned} H \mid \omega\rangle_2 &= \sum_{1\leq j_1<j_2\leq N} (a(j_1+1,j_2)+a(j_1,j_2+1)-a(j_1+1,j_2+1)) \mid j_1 < j_2\rangle_2 \\ &\quad - \sum_{1\leq j\leq N} (a(j+1,j+1)-a(j+1,j+2)) \mid j < j+1\rangle_2. \end{aligned}$$

In order to obey the eigenstate condition, the contact terms in the last line of the above equation should be vanishing:

$$a(j+1,j+1)-a(j+1,j+2)=0.$$

If we test the coefficient as $a(j_1,j_2)=Ce^{i(p_1j_1+p_2j_2)}+De^{i(p_1j_2+p_2j_1)}$, the vanishing contact terms all give

$$\frac{C}{D}=-\frac{x_1}{x_2}.$$

This is certainly consistent with our scattering factor S_{12} in the vacuum equations. The procedure applies to a general Grassmannian.

Geometric basis

The Hamiltonian actually has a geometrical meaning. See Buch and Mihalcea '08. The eigenvalue of this Hamiltonian is the first Schubert class of the $Gr(k; N)$:

$$H \mid \omega\rangle_k = \mathcal{O}_\square \mid \omega\rangle_k$$

where

$$\mathcal{O}_\square := 1 - \prod_{a=1}^{k} x_a .$$

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For example, if k=2 the factor in the above equation is

$$a(j_1+1, j_2) + a(j_1, j_2+1) - a(j_1+1, j_2+1) = (z_1 + z_2 - z_1 z_2)a(j_1, j_2)$$

where the coefficient $(z_1 + z_2 - z_1 z_2) = 1 - x_1 x_2$ (where $z = 1 - x$) is indeed the first Schubert class of $Gr(2, N)$. Thus, one may naturally expect that higher Schubert classes are the eigenvalues of the higher Hamiltonian as well. Quantum K-theory of $Gr(k;N)$ from the integrable model has been investigated by Gorbounov and Korff '2014, although their construction was based on a five-vertex model rather than a spin chain.

Some other insights

- Seiberg duality is the P-symmetry of the associated spin chain
- Different gauge theories, for example, $U(k)$, for different k , can be unified in a single integrable system.
- The Yang-Baxter equation can be derived from the dynamics of BPS domain walls.
- ...

Bosonic chain

There are bosonic integrable systems.. For example, there is a q-deformed bosonic integrable system. The previous study for wave-function was using the algebraic Bethe ansatz. From quantum field theory, we can give a new anstatz and the q-deformed algebra will be emerged from this new ansatz:

$$\psi\left(p_1,\cdots,p_k\right)=A_k\sum_{1\le j_1\le\cdots\le j_k\le n}\sum_{\mathcal{W}\in S_k}S_{\mathcal{W}(a)}e^{\sum_{a=1}^{k}ip_{\mathcal{W}(a)}j_a}|j_1\le\cdots\le j_k\rangle_k.$$

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is very similar to the *coordinate Bethe ansatz* discussed in the spin-$\frac{1}{2}$ chain system except that our case is allowed to put different bosons on the same site. These "contact"-look terms do not merely describe the UV physics of the eigenstates, as they are relevant for IR physics. Then using this wave function, we find a q-deformed algebra emerges from the consistency requirement of the integrable system. This new discovery encourages us that gauge theory can teach us something more about the integrable system.

Conclusion

- We showed how a spin chain emerges from the two-dimensional supersymmetric gauge theory. **Domainwalls play a crucial role**.
- We discussed several examples.

Thanks!