TOWARDS MASTERING DC & AC circuits

① Explore common circuit elements in a DC setting (resistor, capacitor, inductor)

② Gain insight into circuit element behavior in AC setting (phasor language, impedance)

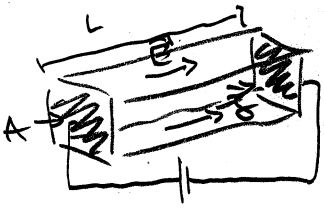
DC circuits: circuits in which DIRECTION of current unchanged

source of current: battery providing a voltage U/potential difference/electro motive force $\mathcal{E}$. "pumps charges against existing potential differences".

RESISTORS: Element made of a material for which "Ohm's law" holds. When potential difference is applied a constant current ensues $I = \Delta U \cdot 1/R$ (R resistance $[R]=\Omega$)

Microscopic picture: Current density $\vec{J} = nq\vec{v}$ $\quad |\vec{J}| = \frac{I}{A}$ induced by presence of an electric field $\boxed{\vec{J} = \sigma \vec{E}}$ (σ conductivity, property of material)

general Ohm's law



if we assume constant $\vec{E}$ inside material

$$\Delta U = E \cdot L \qquad |\vec{J}| = \frac{I}{A} = \sigma \frac{\Delta U}{L} \Rightarrow I = \frac{\Delta U}{L/\sigma A} = \frac{\Delta U}{R}$$

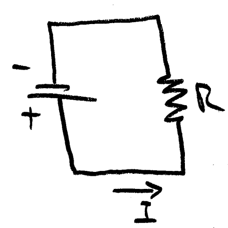
Think about: $\vec{F}_e = q\vec{E}$ if $\vec{E}$ constant ⇒ Newton's 2n law ⇒ $\vec{a}$ = const. ⇒ $\vec{v}(t) = \vec{v}_0 + \vec{a}t$ increases, so why $\vec{J} \propto \vec{v}$ = const?

Reason: electrons in the material bump into each other and the crystal grid → motion becomes randomized. Work done by $\vec{E}$ on charges → converted to heat.

"Resistors dissipate power as heat", "Charges moving through resistor loose some electric potential energy"

How much power? $E_{el} = Q \cdot V$

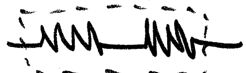
$$\frac{d}{dt}E_{el} = P_{el} = \frac{dQ}{dt} \cdot U = I \cdot V = I^2 R = V^2/R$$



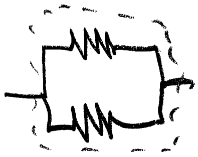
parallel

You are familiar with:

series

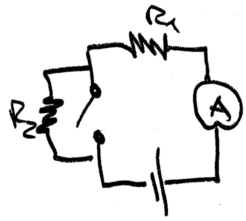


$R_{tot} = R_1 + R_2$

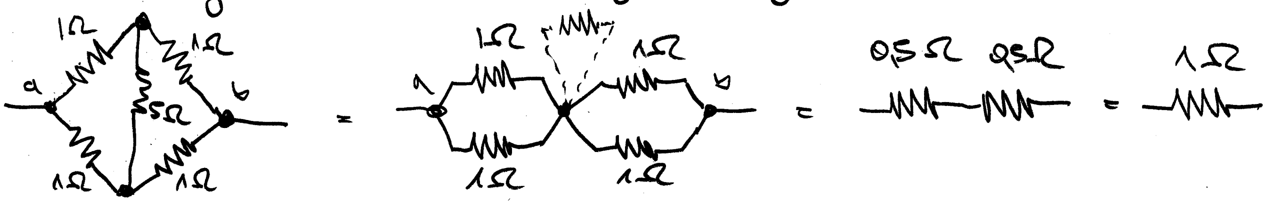


$1/R_{tot} = 1/R_1 + 1/R_2$

Quick refresher: what happens to (A) if switch is open? $I = \Delta U / R_1 + R_2$ if closed $I = \Delta U / R_1$



Reducing circuits with symmetry:



since all resistors 1Ω on outside $I_{ac} = I_{ad} \Rightarrow \Delta U_{ac} = \Delta U_{ad}$

$\Rightarrow \Delta U_{cd} = 0 \Rightarrow I_{cd} = 0$

further study: Thévenin's theorem - Any circuit made solely of resistors and batteries is EQUIVALENT to a single voltage source and a single resistor.

KIRCHHOFF'S RULES: (I) JUNCTION RULE: zero NET current flow into any node $\sum I_{in} = \sum I_{out}$

(due to charge conservation)

(II) LOOP RULE: zero net voltage drop around closed loop

$$\sum_{\text{closed loop}} \Delta V = 0$$

RC Circuits: NEW CIRCUIT ELEMENT "Capacitor"

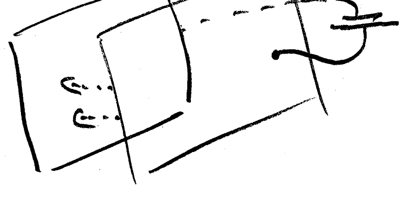
Any two (or more) conductors, which are held at a certain voltage difference produce an electric field, i.e. they carry a certain amount of charge. How much: CAPACITANCE

$\boxed{C = Q/\Delta V}$ For two large parallel plates

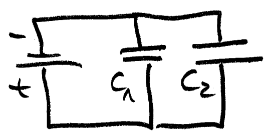
$$C = \frac{\varepsilon_0 A}{d}$$

ε_0: natural constant from Coulomb's Law; A: area of plates; d: distance

$[C] = C/V = F$

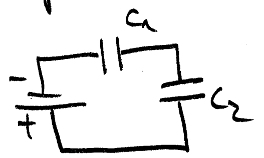


when connected to battery you "fill up" capacitor with charge. Once "full" no more charge flows.



in parallel: total charge is sum of charges

$$Q_{tot} = Q_1 + Q_2$$



in series: charge flow onto C_1 same as on C_2. Total potential difference must be the sum (Kirchhoff)

$$\Delta U_{tot} = \Delta U_1 + \Delta U_2$$

$$\Rightarrow C^{par}_{tot} = \frac{Q_1+Q_2}{\Delta U} = C_1 + C_2$$

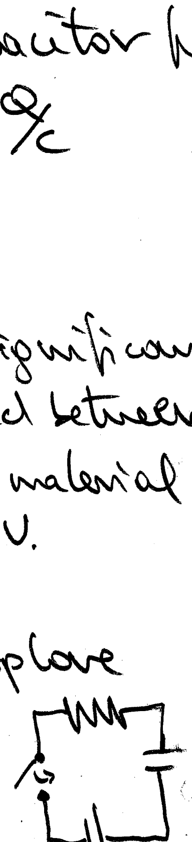
$$\Rightarrow 1/C^{series}_{tot} = \frac{\Delta U}{Q} = \frac{\Delta U_1 + \Delta U_2}{Q} = \frac{1}{C_1} + 1/C_2$$

"Capacitors store electric energy". To "fill up" capacitor from charge 0 to charge Q: $dW = dQ \cdot \Delta U = dQ\, Q/C$

$$W = \int dQ\, Q/C = \frac{1}{2} Q^2_{max}/C = \frac{1}{2} Q_{max} \Delta U_{max} = \frac{1}{2} C (\Delta U_{max})^2$$

"Amount of charge that can be stored increases significantly if a non-conducting (dielectric) material is inserted between capacitor plates" Reason: electric dipoles in the material align and reduce the electric field generated by ΔU.

Now combine CAPACITOR and RESISTOR to explore charging & discharging:



Via Kirchhoff: $\mathcal{E} - \frac{Q(t)}{C} - I(t) R = 0$

- at $t=0$ no charge on capacitor, offers no potential to work against: $\mathcal{E}/R = I(0)$
- when capacitor full no more current and max charge

$$\mathcal{E} = Q_{max}/C \quad \Rightarrow \quad Q_{max} = C\mathcal{E}$$

Now analyze behavior over time: $I = \frac{dQ}{dt} \Rightarrow \frac{dQ}{dt} = \underbrace{\frac{\mathcal{E}}{R}}_{const} - \frac{Q(t)}{RC}$

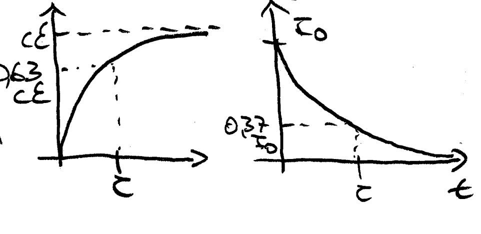
let's solve this: $\frac{dQ}{dt} = \frac{\mathcal{E}C}{RC} - \frac{Q}{RC}$ • move all reference to Q on LHS and all reference to t on RHS

$\Leftrightarrow \frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{RC}$ • integrate $\int_0^{Q(t)} \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC}$ • substitute $y = \mathcal{E}C - Q$

$$-\left(\log(\mathcal{E}C - Q) - \log(\mathcal{E}C)\right) = \frac{1}{RC}(t-0) \Rightarrow \log\left(\frac{\mathcal{E}C - Q}{\mathcal{E}C}\right) = -\frac{t}{RC}$$

$$\frac{Q - \mathcal{E}C}{-\mathcal{E}C} = \exp[-t/RC] \Rightarrow \boxed{Q(t) = C\mathcal{E} - C\mathcal{E}e^{-t/RC} = C\mathcal{E}(1 - e^{-t/RC})}$$

accompanying current $I = \frac{dQ}{dt} = \frac{C\mathcal{E}}{RC} e^{-t/RC}$

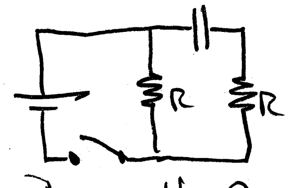


Time constant: $\tau = RC$ "how fast does it fill up"

$[RC] = \Omega \cdot F = V/A \cdot C/V = C/A = C/C/s = s$ ✓

Discharging: remove battery from circuit and let charge flow between capacitor plates: $-Q/C - IR = 0 \Rightarrow Q(t) = Q_{max} e^{-t/RC}$

$I(t) = -Q_0/RC \exp(-t/RC)$



Quick check: ① just after switch closes what current flows in the battery? since no Q in capacitor $\Delta V_C = 0$ via Kirchhoff $\Delta V_{R_1} = \Delta V_{R_2} = \mathcal{E}$ ⇒ resistors in parallel

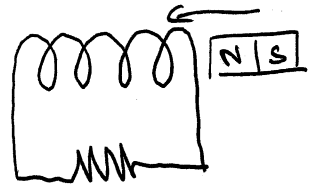
$1/R_{tot} = 1/R_1 + 1/R_2 = 2/R \Rightarrow I = \mathcal{E}/R_{tot} = 2\mathcal{E}/R$

② when capacitor filled up: no current through outer loop
$I = \mathcal{E}/R$.

FINAL ELEMENT: INDUCTOR (explicitly solenoid)

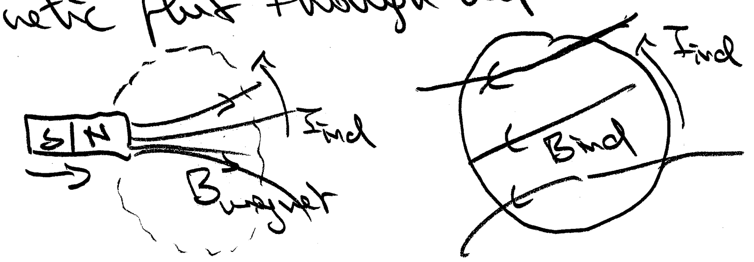
Closely connected to Faraday's law of INDUCTION: An electromotive force $\mathcal{E}_{ind}$ is induced by time varying magnetic flux: $\mathcal{E} = -\frac{d\Phi_B}{dt}$ $\quad \Phi_B = \vec{B} \cdot \vec{A}$ (either $\vec{B}$ changes or $\vec{A}$ changes)

$\Phi_B = B \cdot A \cdot \cos\theta$



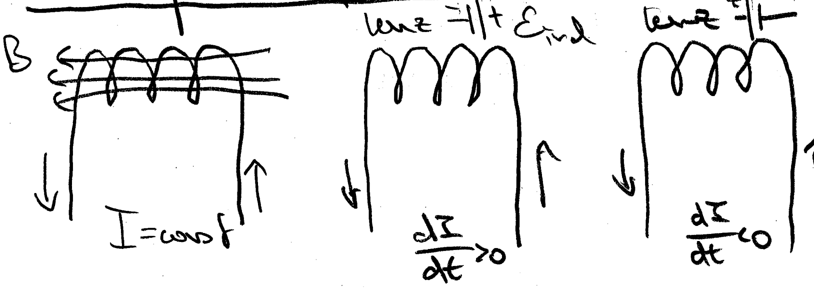
induces a current $I = \frac{\mathcal{E}_{ind}}{R}$
what is direction of current?

Lenz's law: Induced current in the loop is in the direction that creates a magnetic field that opposes the change in magnetic flux through loop.



Necessary consequence: If a current is induced in a conductor there must be an electric field present that moves the charges.

Consequence for circuits: SELF INDUCTANCE

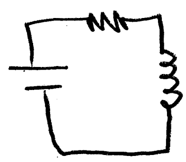


$\mathcal{E}_L = -L\frac{dI}{dt}$ ← inductance

with Faraday: $\mathcal{E}_L = -N\frac{d\Phi_B}{dt}$ ← # of loop in solenoid

$\Rightarrow L_{solenoid} = \frac{N\Phi_B}{I}$ $[L] = H$ "Henry"

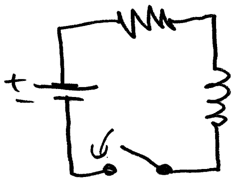
"Solenoid opposes changes in the current"



Energy associated with magn. field in solenoid:

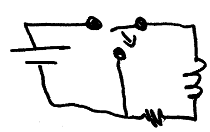
$\mathcal{E} - IR - L\frac{dI}{dt} = 0$ multiply by I $\underbrace{I\mathcal{E}}_{\text{power supplied by battery}} = \underbrace{I^2R}_{\text{power dissipated by R}} - LI\frac{dI}{dt}$ ← power assoc with ind.

$\frac{dE}{dt} = LI\frac{dI}{dt}$ $\boxed{E = \int dE = \int_0^I LI\,dI = \frac{1}{2}LI^2}$



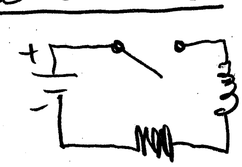
RL circuit: when switched on similar analysis as for RC circuit for Q, here for I

$I(t) = \mathcal{E}/R\,(1-e^{-t\cdot R/L})$ time constant $\tau = L/R$

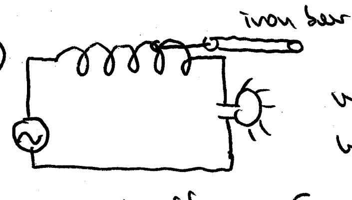


switching off: $I(t) = \mathcal{E}/R\; e^{-t/\tau}$

Cross-checks: (1) just after closing switch, across which circuit element is the voltage equal to the battery emf. (2) after a long time?

(I) 

(A1) t=0 no current $\Delta U_R = 0$ so with Kirchhoff $\Delta U_S = \mathcal{E}$ (A2) $t\to\infty$ I = const in solenoid so $\mathcal{E}_{ind} = 0$ $\Rightarrow \Delta U_R = \mathcal{E}$.

(II) 

Assume sinusoidal power source. When inserting metal bar B field inside solenoid increases, L increases. What happens to brightness of bulb?

Kirchhoff: $\mathcal{E} - L\frac{dI}{dt} - IR = 0$ → since L increases more voltage drop over solenoid, less drop over lamp → dimmer

Oscillations in an LC circuit: electromagnetic analog

to a block on a spring on frictionless surface. (Electric energy $\equiv$ potential energy and magnetic energy $\equiv$ kinetic energy)

Energy accounting $E_C = \frac{1}{2} Q^2/C$ $\quad E_L = \frac{1}{2} L I^2$

Energy conservation: $0 \doteq \frac{d}{dt} E_{tot} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2} L I^2 \right) = \frac{Q}{C}\frac{dQ}{dt} + L I \frac{dI}{dt}$

since $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$ $\quad \Rightarrow \boxed{\frac{d^2Q}{dt^2} = -\frac{1}{LC} Q(t)}$

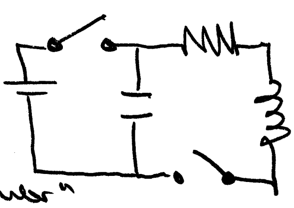
same form as $\frac{d^2x}{dt^2} = -\frac{k}{m} x = -\omega^2 x$ "oscillations in amount of charge on capacitor plates with angular frequency $\boxed{\omega = \sqrt{1/LC}}$

Solution requires INITIAL CONDITIONS: e.g. fully charged capacitor: $Q(0) = Q_{max}$ $\quad I(0) = 0$ $\Rightarrow Q(t) = Q_{max}\cos(\omega t)$

$I(t) = -Q_{max}\,\omega \sin(\omega t)$ $\quad$ Sanity check:

$$E_{tot} = \frac{1}{2C} Q_{max}^2 \cos^2(\omega t) + \frac{1}{2} L Q_{max}\, \omega^2 \sin^2(\omega t) = \frac{Q_{max}}{2C}(\cos^2 + \sin^2) = \frac{Q_{max}}{2C}$$

RLC circuit "Damped harmonic oscillations"



Energy conservation $\frac{d}{dt}(E_C + E_L + E_R) \doteq 0$ $\quad \frac{dE_R}{dt} = -I^2 R$ "dissipate power"

$$L I \frac{d^2Q}{dt^2} + I^2 R + \frac{Q}{C} I = 0$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad \Leftrightarrow \quad m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \text{ANALOGOUS}$$

for small R: $Q(t) = Q_{max}\, e^{-\frac{R}{2L}t} \cos(\omega_d t)$ $\quad \omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$

AC Circuits

Power source produces $V(t) = V_{max} \cdot \cos(\Omega t + \phi)$

Use of complex numbers allows us to deploy similar techniques as for DC circuits.

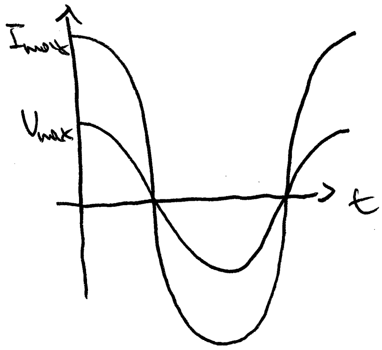
Promote to complex values $V(t) \mapsto \tilde{V}(t) = \tilde{V}_0\, e^{i\Omega t}$ $\quad \tilde{V}_0 = V_{max}\, e^{i\phi}$

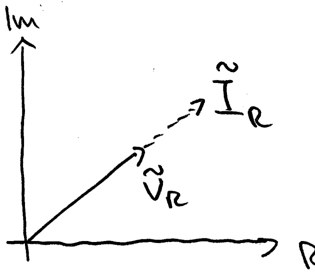
Physical voltage described by its REAL PART $V(t) = \mathrm{Re}[\tilde{V}(t)]$

(I) Resistors in AC circuits

Its voltage drop follows that of the source instantaneously i.e. voltage and current are IN PHASE

$V_R(t) = U_{max} \cos(\Omega t + \phi)$ $\quad R = U/I \quad \Rightarrow \quad I_R(t) = U_{max}/R \cos(\Omega t + \phi)$





PHASOR DIAGRAM

"complex Ohm's law"

$$\tilde{V}_R = Z_R \cdot \tilde{I}_R$$

here $Z_R = R$ purely real

"Z complex IMPEDANCE"

instead of I_{max} often quoted root-mean-square current: average current over one cycle $I_{rms} = \sqrt{(\overline{I^2})}$

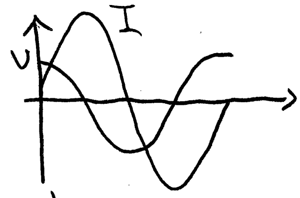
with $(\overline{I^2}) = \int_0^T dt\, I_{max}^2 \sin^2(\Omega t) = \frac{1}{2} I_{max}^2$ via $\sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$

$I_{rms} = I_{max}/\sqrt{2} = 0.707\, I_{max}$

(II) Inductors in AC circuits

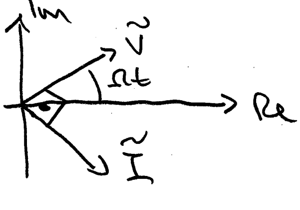
Via Kirchhoff $V(t) - L\frac{dI}{dt} = 0$

i.e. if $V(t)$ is a cosine $I_s(t)$ is a sine.



"Solenoids oppose changes in current", so that current lags voltage by $\phi = \pi/2$

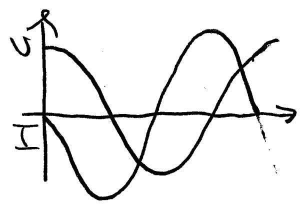
implemented via $\boxed{Z_I = i\Omega L}$



(III) Capacitors in AC circuits

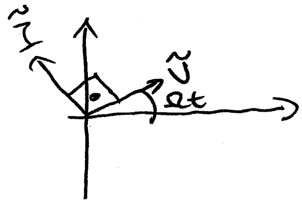
With Kirchhoff $V + Q/C = 0 \quad \Rightarrow \quad Q(t) = C\, U_{max} \cos(\Omega t + \phi)$

$\Rightarrow \quad I = \frac{dQ}{dt} = C U_{max} \Omega \sin(\Omega t + \phi)$

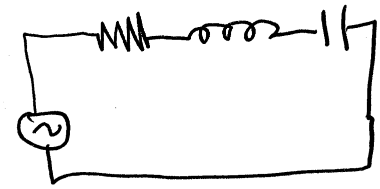


"charges must flow onto capacitor for voltage to arise". Current ahead of voltage by $\pi/2$

$$\boxed{Z_C = \frac{1}{i\Omega C}}$$



RLC series circuit



Since in series: CURRENT at all points must have the same amplitude and phase

$\Rightarrow$ Voltage across elements differs

With Kirchhoff: $\tilde{V}(t) = V_{max}\, e^{i\Omega t}$

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = U_{max}\, e^{i\Omega t}$$

ANALOG to driven and damped harmonic mechanical oscillator.

Steady state solution will oscillate with external Ω.

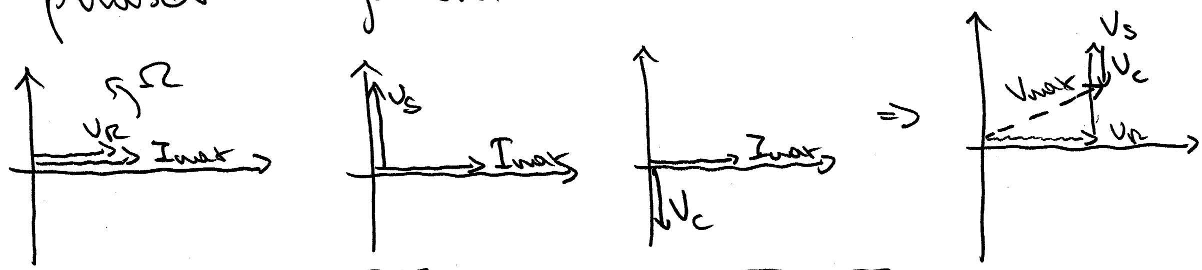
Ansatz: $I(t) = I_0\, e^{i\Omega t} \Rightarrow \dot{I}(t) = i\Omega I_0\, e^{i\Omega t}$ $\quad Q(t) = \frac{1}{i\Omega} I_0 e^{i\Omega t}$

$$\left(i\Omega L + R + \frac{1}{i\Omega C}\right) I_{max} = V_{max} \Rightarrow |I_{max}| = \frac{|U_{max}|}{|i\Omega L + R + \frac{1}{i\Omega C}|}$$

$$\Rightarrow |I_{max}| = \frac{|U_{max}|}{\sqrt{R^2 + (\Omega L - \frac{1}{\Omega C})}}$$

Maximum amplitude = RESONANCE for $\Omega = 1/\sqrt{LC}$

In phasor diagram:



$$V_{max} = \sqrt{V_R^2 + (U_L - V_C)^2} = I_{max}\sqrt{R^2 + (\Omega L - \frac{1}{\Omega C})^2}$$

The smaller the resistance the sharper the resonance peak in I_{max} as $\Omega \to 1/\sqrt{LC}$