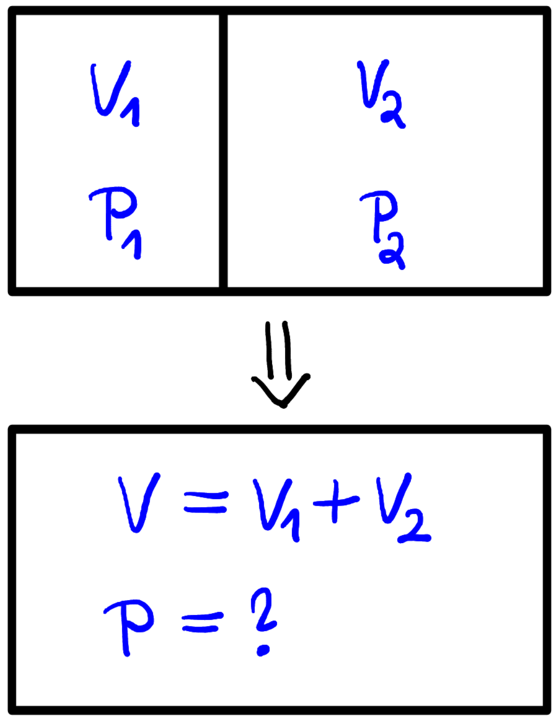
PROBLEM : two boxes with volume (pressure) $V_1 (P_1)$ and $V_2 (P_2)$ contain ideal gases of equal temperature. Find the pressure in the system after the boxes are connected and the gases are allowed to mix



Solution : write

$$\left.\begin{aligned} P_1 V_1 &= N_1 k_B T \\ P_2 V_2 &= N_2 k_B T \end{aligned}\right\} \text{ before}$$

$$\Big\downarrow \text{add}$$

$$P_1 V_1 + P_2 V_2 = (N_1 + N_2) k_B T \ldots \text{ compare to } P(V_1 + V_2) = (N_1 + N_2) k_B T$$

Final result :

$$P = \frac{P_1 V_1 + P_2 V_2}{V_1 + V_2}$$

PROBLEM : estimate the mass of the Earth's atmosphere

Solution 1 : from pressure on surface

$$Mg \overset{?}{=} 4\pi r_0^2 P_0 \longrightarrow M = \frac{4\pi r_0^2 P_0}{g} \approx 5.27\times 10^{18}\,\mathrm{kg}$$

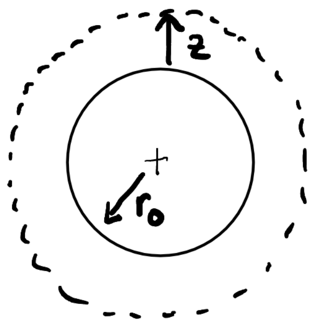
r_0: 6371 km P_0: 1atm = 101 325 Pa

Solution 2 : from the barometric formula $dP = -\rho g\,dz = -\frac{P}{RT_0} g\,dz$

R: 287 J/kg·K T_0: global mean temp. at sealevel (288K)

this gives $P(z) = P_0 \exp\left(-\frac{gz}{RT_0}\right) \Rightarrow \rho(z) = \rho_0 \exp\left(-\frac{gz}{RT_0}\right)$

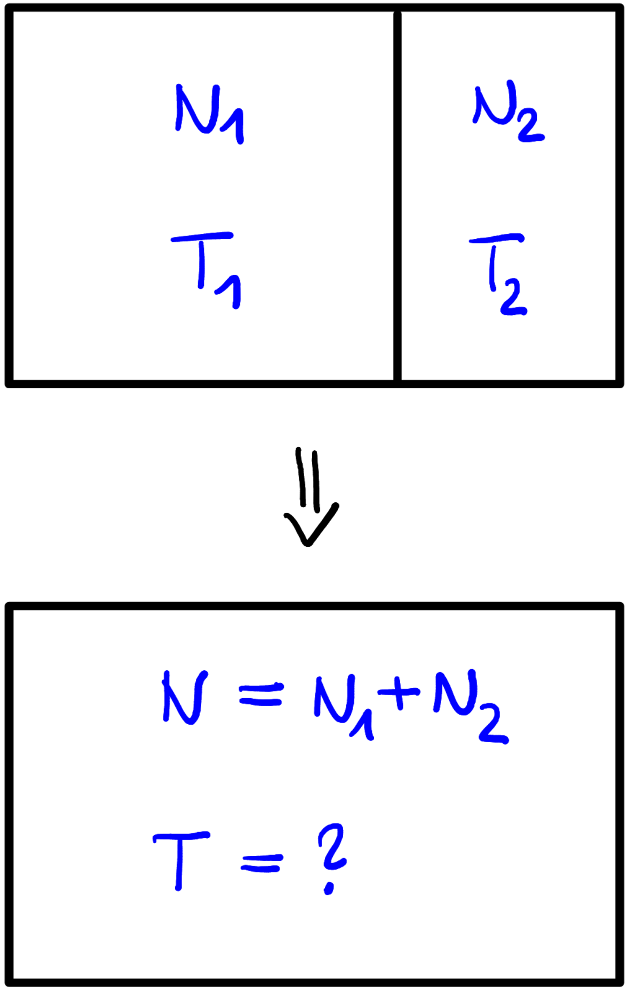
Now get the total mass by integration of density over volume :



$$M = \int_0^\infty 4\pi (r_0+z)^2 \rho(z)\,dz \approx 5.28\times 10^{18}\,\mathrm{kg}$$

very close to the previous naive solution

PROBLEM : mix two ideal monoatomic gases with N_1 (N_2) molecules and temperatures T_1 (T_2). What is the final temperature after mixing?



Solution : use conservation of (internal) energy

$$U_1 = \frac{3}{2} N_1 k_B T_1 \quad \& \quad U_2 = \frac{3}{2} N_2 k_B T_2$$

$$\Downarrow$$

$$U = U_1 + U_2 = \frac{3}{2} k_B (N_1 T_1 + N_2 T_2) \stackrel{?}{=} \frac{3}{2} k_B (N_1 + N_2) T$$

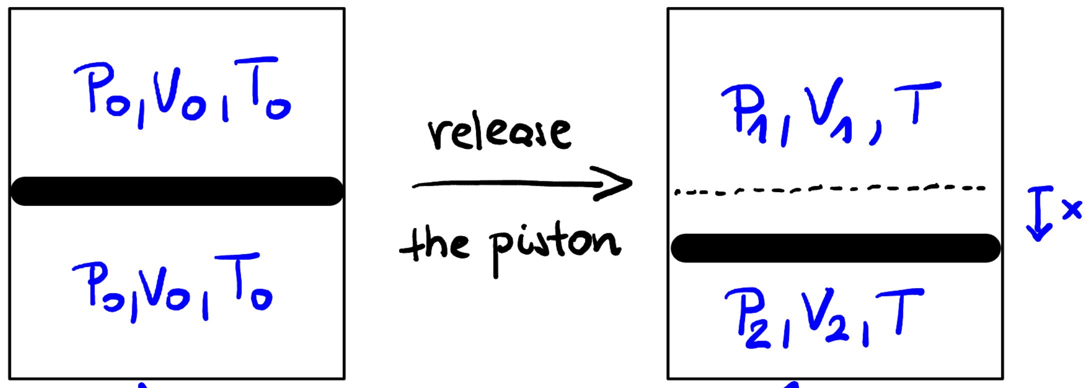
Final result :
$$T = \frac{N_1 T_1 + N_2 T_2}{N_1 + N_2}$$

PROBLEM (CZPhO 2015)

Given : $P_0 = 100\,\text{kPa}$, $T_0 = 300\,\text{K}$

$P_2 - P_1 = 10\,\text{kPa}$, $c_v = \frac{5}{2}R$

Task: find the change in temperature $\Delta T = T - T_0$



heavy piston, thermally conducting, initially fixed

thermally insulating walls

Solution : denote the area and mass of the piston as $\underline{A}$, $\underline{m}$

In the new equilibrium, $\Delta P = P_2 - P_1 = \frac{mg}{A}$

The internal energy of the gas has increased by $\Delta U = mgx = \Delta P A x$

At the same time, $\Delta U = \frac{5}{2} N k_B \Delta T = \frac{5}{2} \times 2 \frac{P_0 V_0}{T_0} \Delta T = 5 P_0 V_0 \frac{\Delta T}{T_0}$

$$Ax = \frac{5 P_0 V_0}{\Delta P} \frac{\Delta T}{T_0}$$

Now calculate the pressure difference using the ideal gas EoS:

$$\left.\begin{aligned} P_1 &= \frac{P_0 V_0}{T_0}\frac{T}{V_1} \\ P_2 &= \frac{P_0 V_0}{T_0}\frac{T}{V_2} \end{aligned}\right\} \longrightarrow \Delta P = \frac{P_0 V_0}{T_0} T \left(\frac{1}{V_2} - \frac{1}{V_1}\right) = \frac{P_0 V_0}{T_0} T \left(\frac{1}{V_0 - Ax} - \frac{1}{V_0 + Ax}\right)$$

$$= P_0 V_0 \underbrace{\frac{T}{T_0}}_{= 1 + \frac{\Delta T}{T_0}} \frac{2Ax}{V_0^2 - (Ax)^2}$$

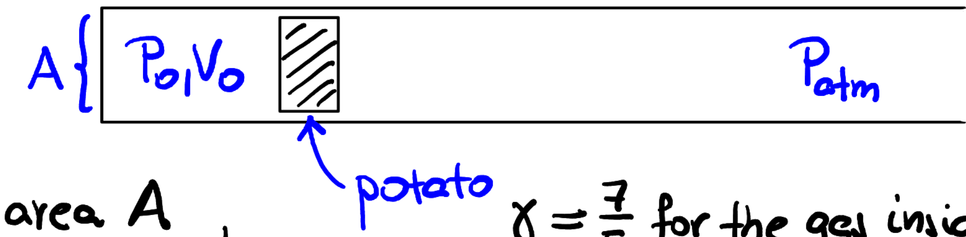
Put this together with the previously found expression for Ax:

$$\frac{\Delta P}{P_0} = \left(1 + \frac{\Delta T}{T_0}\right) \frac{10 \frac{P_0}{\Delta P}\frac{\Delta T}{T_0}}{1 - 25\left(\frac{P_0}{\Delta P}\frac{\Delta T}{T_0}\right)^2} \Rightarrow \underline{\underline{35\left(\frac{\Delta T}{T_0}\right)^2 + 10\frac{\Delta T}{T_0} - \left(\frac{\Delta P}{P_0}\right)^2 = 0}}$$

This is a quadratic equation for $\frac{\Delta T}{T_0}$ with a single positive root:

$$\frac{\Delta T}{T_0} = \frac{1}{70}\left[-10 + \sqrt{100 + 140\left(\frac{\Delta P}{P_0}\right)^2}\right] \approx 0.001 \Rightarrow \underline{\underline{\Delta T \approx 0.3\,\mathrm{K}}}$$

PROBLEM (USPhO 2009) : Potato gun



Given : initial pressure P_0 & volume V_0, area A, $\gamma = \frac{7}{5}$ for the gas inside gun

Task : find length of the gun at which the exit speed of the potato is maximal, and the corresponding kinetic energy of the potato

Solution : maximum speed when final pressure inside the gun equals P_{atm}

$$P_0 V_0^{\gamma} = P_{atm} V^{\gamma} \Rightarrow V = V_0 \left(\frac{P_0}{P_{atm}}\right)^{1/\gamma} \Rightarrow L = \frac{V_0}{A}\left(\frac{P_0}{P_{atm}}\right)^{\gamma}$$

Kinetic energy of the potato = work done by the gas − work done by P_{atm}

$$E_{kin} = W_{out} - P_{atm}(V - V_0) = \frac{P_0 V_0 - P_{atm} V}{\gamma - 1} - P_{atm}(V - V_0) = V_0\left[\frac{P_0}{\gamma-1} + P_{atm} - \frac{\gamma}{\gamma-1} P_{atm}\left(\frac{P_0}{P_{atm}}\right)^{1/\gamma}\right]$$

$$E_{kin} = \left(\frac{5}{2}P_0 + P_{atm} - \frac{7}{2} P_0^{5/7} P_{atm}^{2/7}\right) V_0$$

PROBLEM : What is the amount of water in the air in an office of $10\,m^2$ and $3\,m$ ceiling if the temperature is $22°C$ and $\varphi = 17\%$?

Solution :

Calculate the partial pressure :

$$\left.\begin{array}{r} P_{sat}(22°C) = 2646\,Pa \\ \varphi = 17\% \end{array}\right\} \longrightarrow \underline{P \approx 450\,Pa}$$

Gas constant of water : $R = 461.5\, J/kg \cdot K \longrightarrow \rho = \dfrac{P}{RT} \approx 3.30\, g/m^3$

$$\downarrow$$

$$M = \rho V = 3.30\, g/m^3 \times 30\, m^3 \approx \underline{\underline{99\,g}}$$

PROBLEM : suppose the outside air has temperature 20°C and humidity 50%.

How does the humidity change if the temperature increases/decreases by 10°C?

Solution : assume for simplicity that the absolute pressure of the water vapor in the air does not change as the temperature increases.

T [°C]	P_{sat} [kPa]
10	1.227
20	2.340
30	4.248

temperature increase :

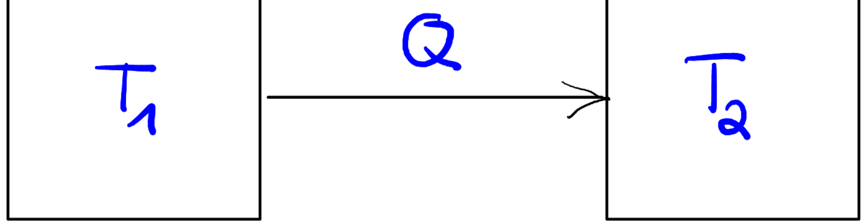
$$\varphi = 50\% \times \frac{2.340}{4.248} \approx \underline{\underline{28\%}}$$

temperature decrease :

$$\varphi = 50\% \times \frac{2.340}{1.227} \approx \underline{\underline{95\%}}$$

PROBLEM : prove that heat cannot spontaneously flow from a colder object to a hotter object

Solution : treat $T_1 \cup T_2$ as an isolated system



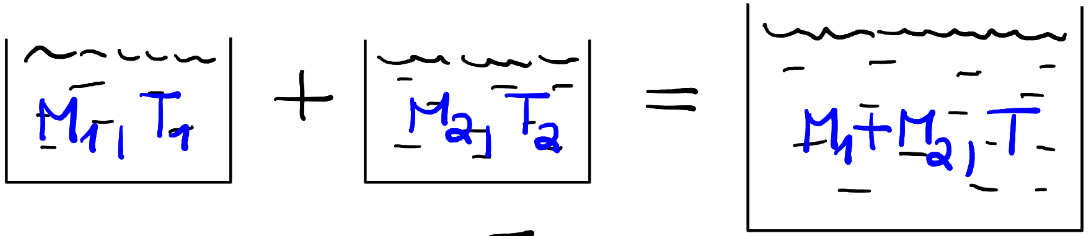
$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{Q}{T_2} - \frac{Q}{T_1}$$

$$= Q\left(\frac{1}{T_2} - \frac{1}{T_1}\right) \overset{?}{\geq} 0 \implies \underline{\underline{T_2 \leq T_1}}$$

Heat always flows from a high-temperature region to a low-temperature one.

PROBLEM : entropy involved in mixing of water of different temperatures

Solution : first find the final temperature using energy conservation (1^{st} law)



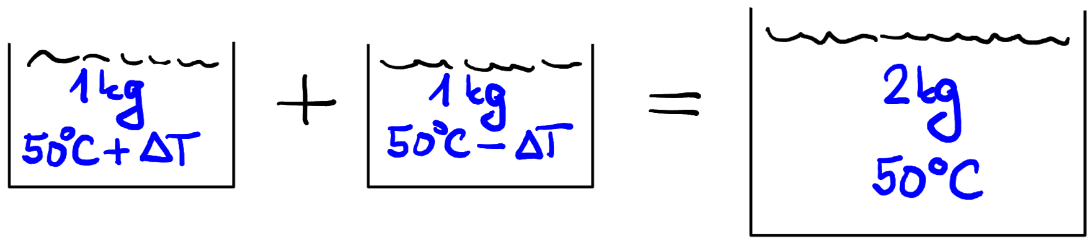
$$\Delta U = c_v M_1 (T - T_1) + c_v M_2 (T - T_2) \Rightarrow \quad \underline{\underline{T = \frac{M_1 T_1 + M_2 T_2}{M_1 + M_2}}}$$

Now calculate the change in entropy, assuming slow (quasiequilibrium mixing):

$$\Delta S_1 = \int \frac{dQ_1}{T_1} = \int_{T_1}^{T} \frac{M_1 c_v \, dT_1}{T_1} = M_1 c_v \ln \frac{T}{T_1}$$

$$\Delta S = \Delta S_1 + \Delta S_2 = \underline{\underline{c_v \left(M_1 \ln \frac{T}{T_1} + M_2 \ln \frac{T}{T_2} \right)}}$$

How big is the entropy change numerically? Take a specific example:



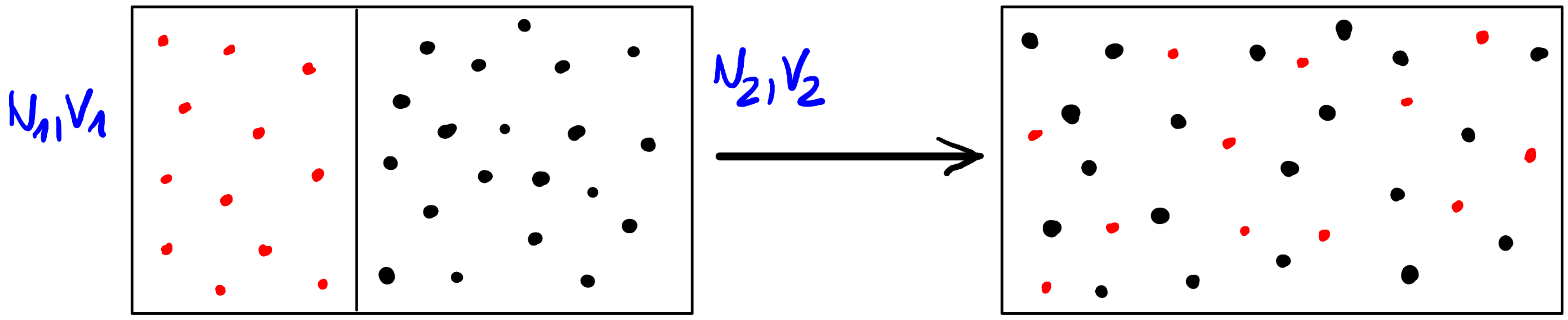
$M_1 = M_2 = 1\,\mathrm{kg}$

$c_v \approx 4.2\,\mathrm{kJ/kg \cdot K}$

ΔT [°C]	1	2	5	10	20	50
ΔS [J/K]	0.04	0.16	1.0	4.0	16	102

- amount of heat exchanged grows as ΔT
- entropy increase grows as ΔT^2

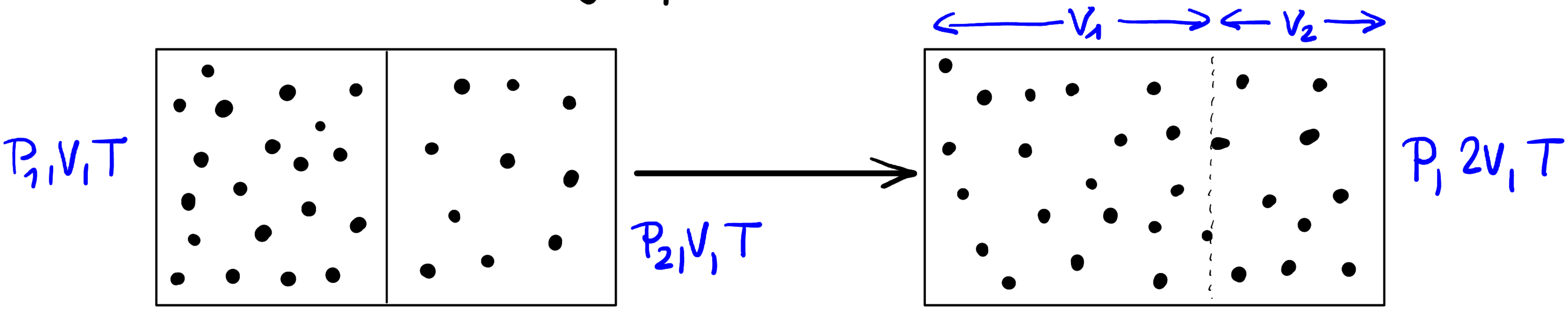
<u>PROBLEM</u> : Calculate the change of entropy involved in mixing of two different ideal gases of equal temperatures



Treat the process as independent free expansion of both gases!

$$\Delta S = N_1 k_B \ln \frac{V_1 + V_2}{V_1} + N_2 k_B \ln \frac{V_1 + V_2}{V_2}$$

PROBLEM : the same problem as before but now with identical gases, and for simplicity equal initial volumes.



We know from before that the final pressure is $P = \frac{1}{2}(P_1 + P_2)$. Think of the equilibration process as isothermal expansion to such volumes V_1, V_2 that the pressures become equal :

$$V_1 = \frac{P_1 V}{P} = V \frac{2P_1}{P_1 + P_2}, \quad V_2 = \frac{P_2 V}{P} = V \frac{2P_2}{P_1 + P_2}.$$

This gives

$$\underline{\underline{\Delta S = \frac{P_1 V}{T} \ln \frac{2P_1}{P_1 + P_2} + \frac{P_2 V}{T} \ln \frac{2P_2}{P_1 + P_2}}}$$