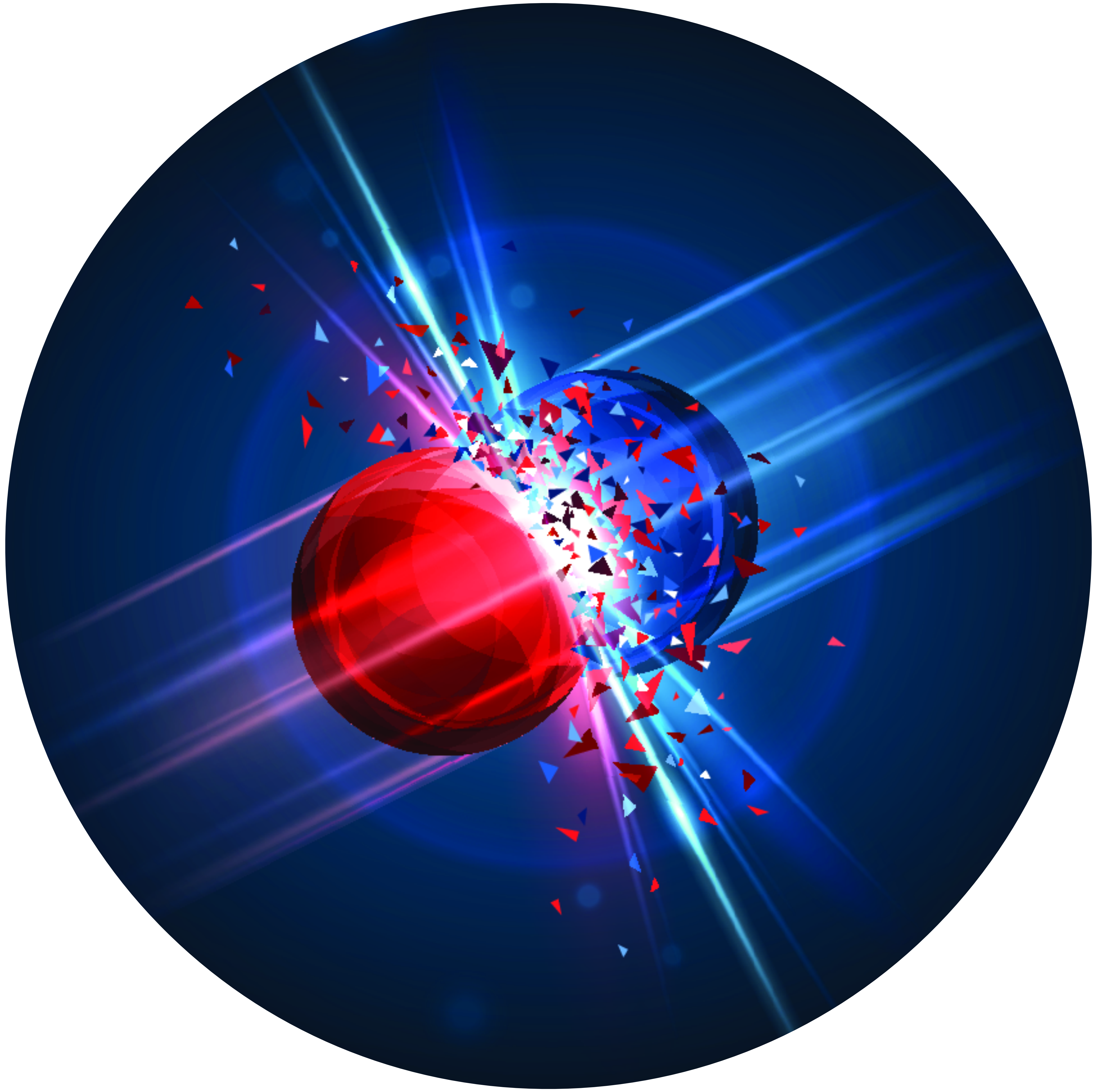
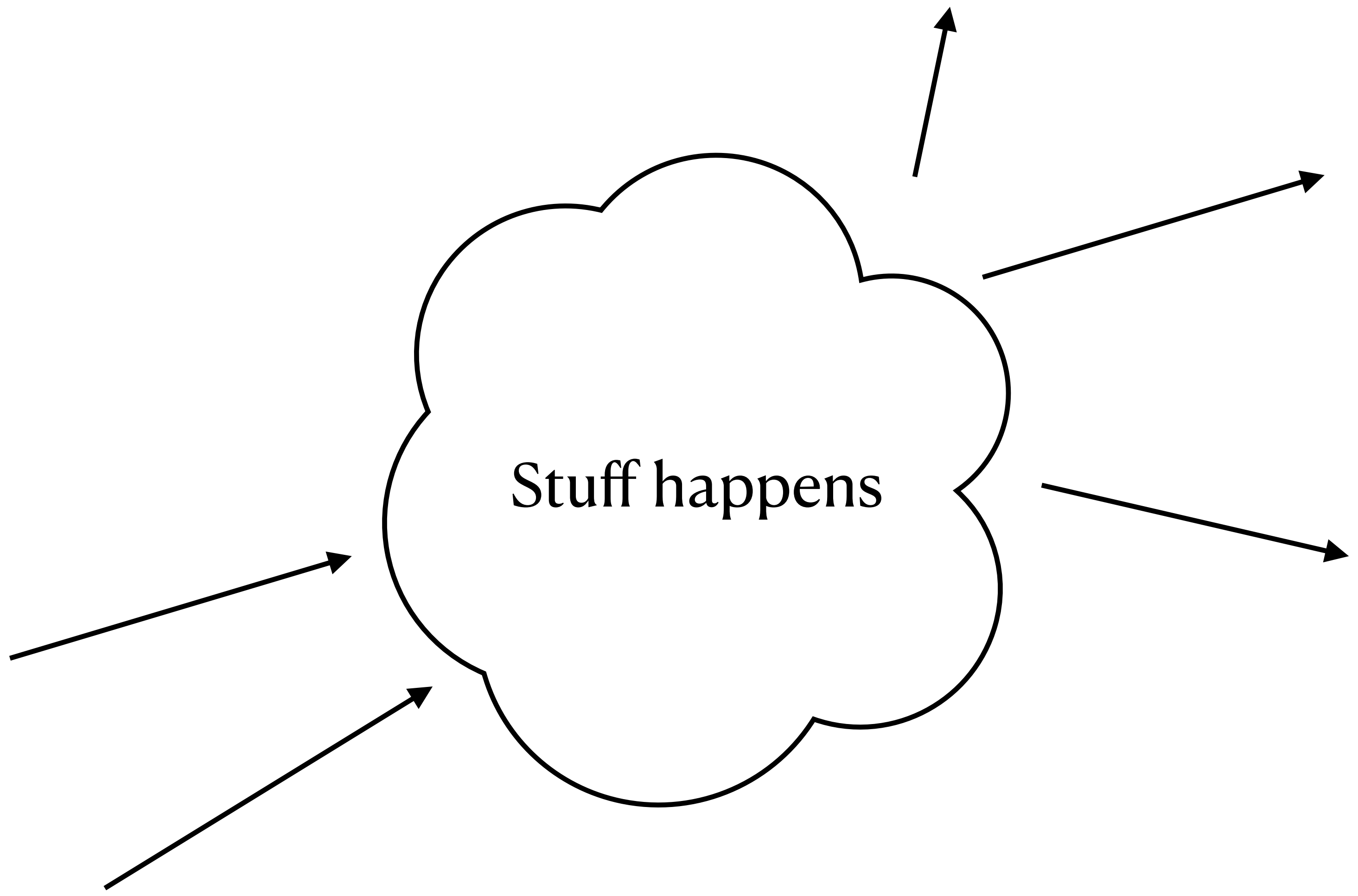


Geometry of Scattering Amplitudes

Andreas Helset, CERN

Scattering Amplitudes



Scattering matrix, "S-matrix"

$$\hat{S}: \quad |\psi_{\text{in}}\rangle \to |\psi_{\text{out}}\rangle = \hat{S}|\psi_{\text{in}}\rangle$$

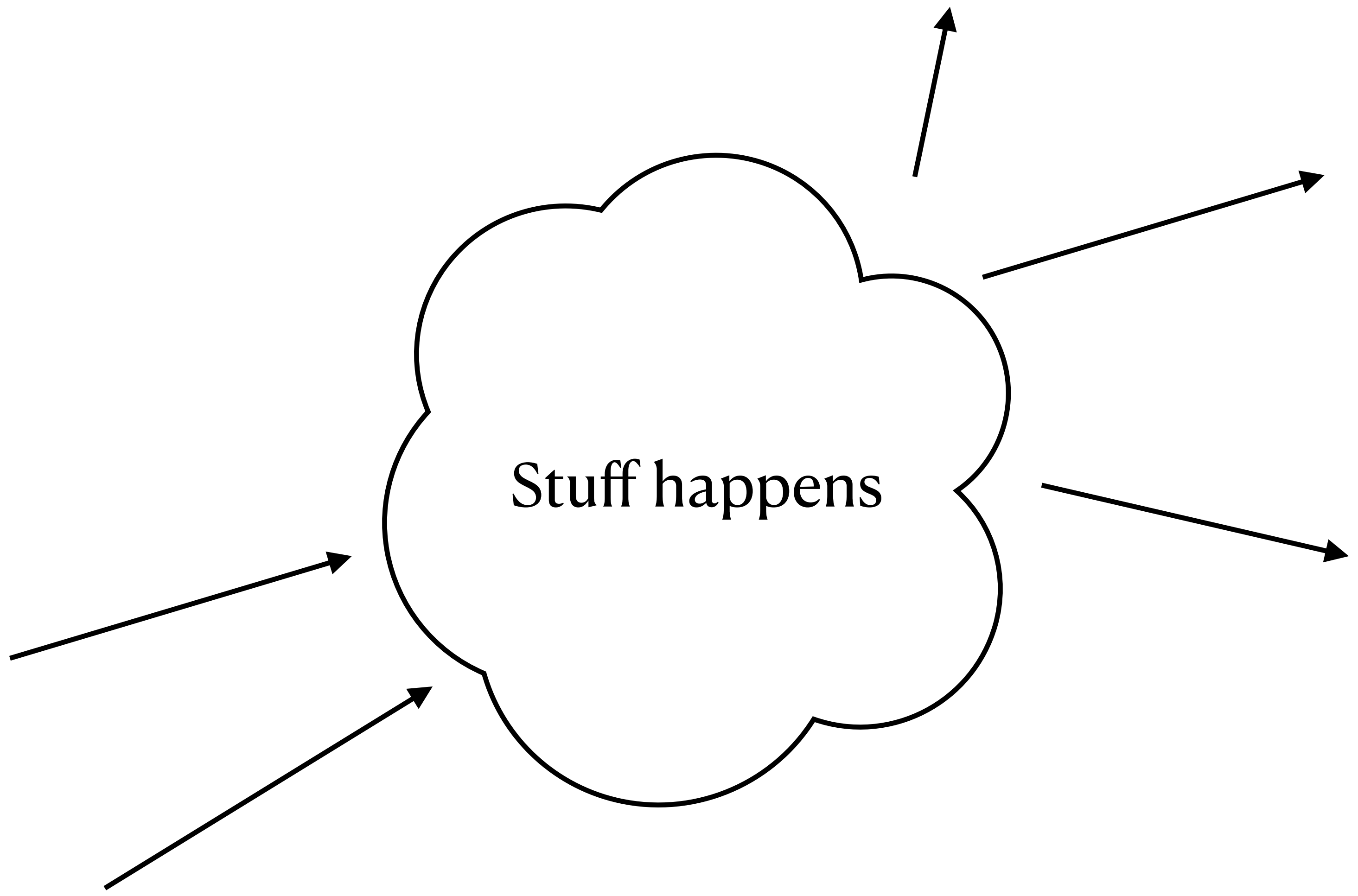
Key object for collider experiments

$$\frac{d\sigma}{d\Omega} \sim |\mathcal{A}|^2$$

Unitarity

$$\hat{S}\hat{S}^\dagger = 1$$

Scattering Amplitudes



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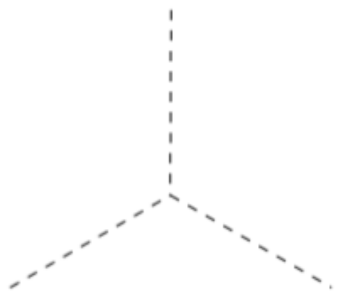
Key object for collider experiments

$$\frac{d\sigma}{d\Omega} \sim |\mathcal{A}|^2$$

QFT 1:

Lagrangian —> Feynman Rules —> Scattering Amplitude

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\phi^3$$

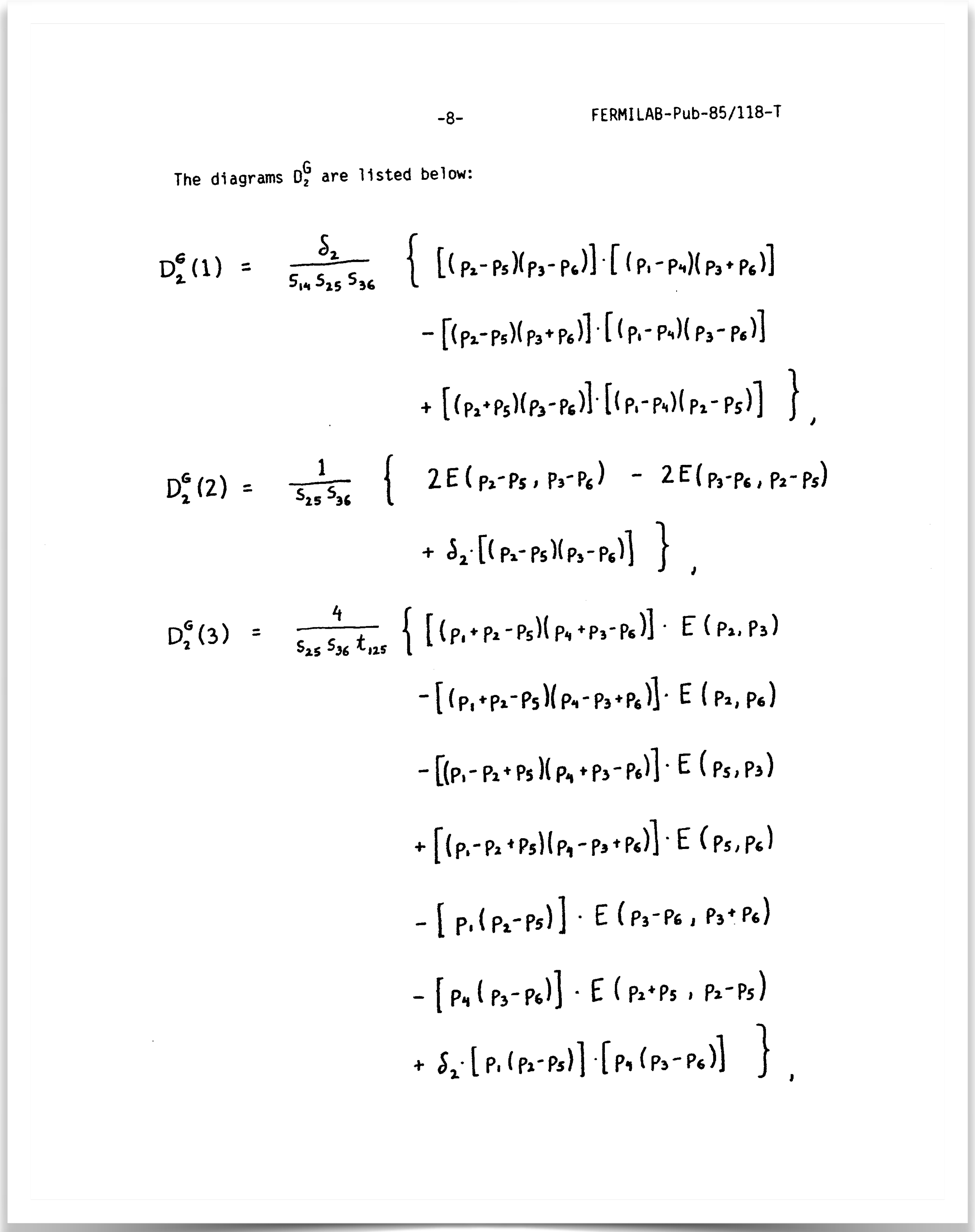


$$\mathcal{A} = \lambda$$

Scattering Amplitudes

Surprising simplicity!

Gluon amplitude from Feynman diagrams



The diagrams D_2^G are listed below:

$$D_2^G(1) = \frac{\delta_2}{s_{14}s_{25}s_{36}} \Big\{ [(p_2-p_5)(p_3-p_6)]\cdot[(p_1-p_4)(p_3+p_6)] - [(p_2-p_5)(p_3+p_6)]\cdot[(p_1-p_4)(p_3-p_6)] + [(p_2+p_5)(p_3-p_6)]\cdot[(p_1-p_4)(p_2-p_5)] \Big\},$$

$$D_2^G(2) = \frac{1}{s_{25}s_{36}} \Big\{ 2E(p_2-p_5, p_3-p_6) - 2E(p_3-p_6, p_2-p_5) + \delta_2\cdot[(p_2-p_5)(p_3-p_6)] \Big\},$$

$$D_2^G(3) = \frac{4}{s_{25}s_{36}t_{125}} \Big\{ [(p_1+p_2-p_5)(p_4+p_3-p_6)]\cdot E(p_2,p_3) - [(p_1+p_2-p_5)(p_4-p_3+p_6)]\cdot E(p_2,p_6) - [(p_1-p_2+p_5)(p_4+p_3-p_6)]\cdot E(p_5,p_3) + [(p_1-p_2+p_5)(p_4-p_3+p_6)]\cdot E(p_5,p_6) - [p_1(p_2-p_5)]\cdot E(p_3-p_6, p_3+p_6) - [p_4(p_3-p_6)]\cdot E(p_2+p_5, p_2-p_5) + \delta_2\cdot[p_1(p_2-p_5)]\cdot[p_4(p_3-p_6)] \Big\},$$

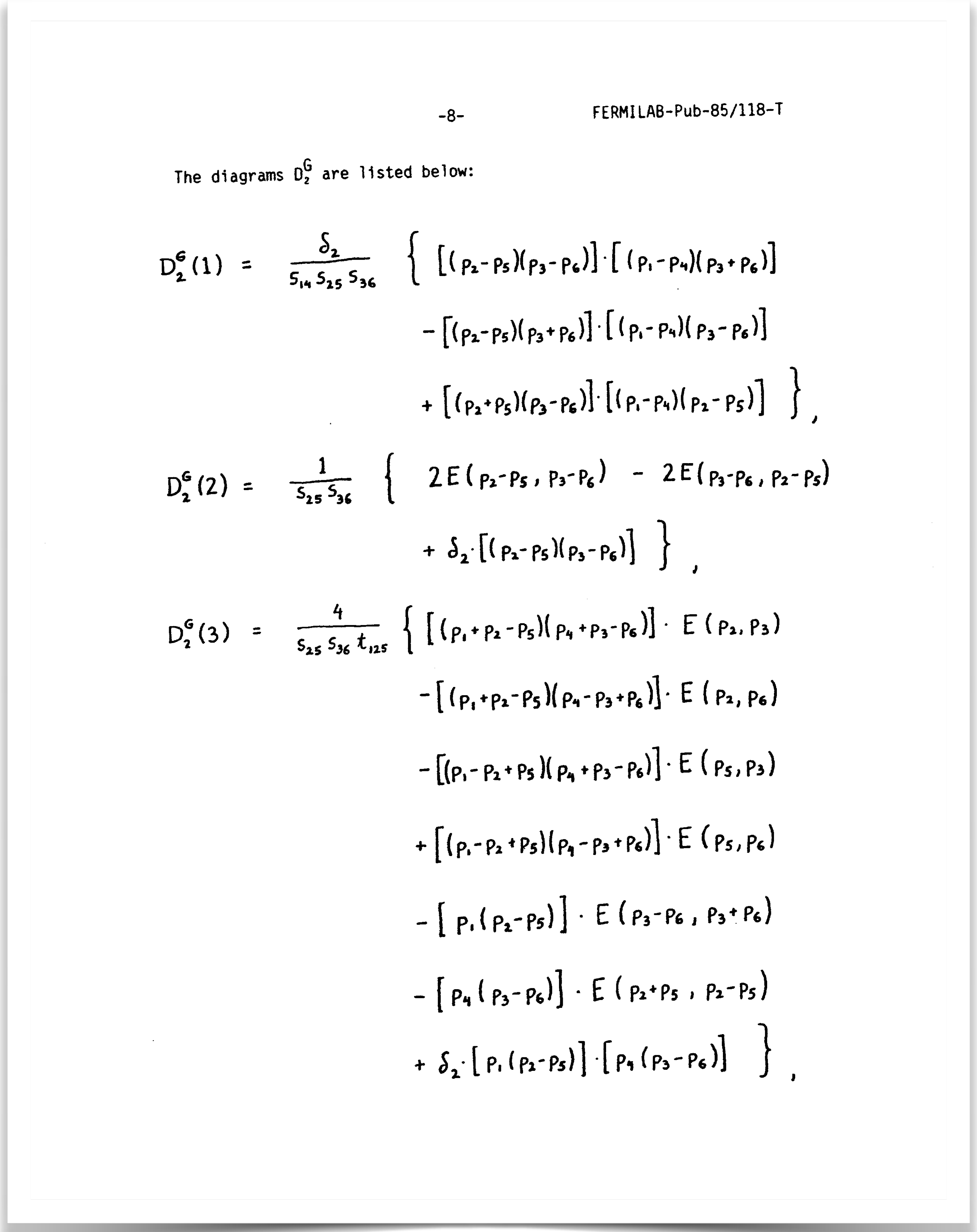
Parke, Taylor '86

… + 8 more pages

Scattering Amplitudes

Surprising simplicity!

Gluon amplitude from Feynman diagrams



-8- FERMILAB-Pub-85/118-T

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Parke, Taylor '86

... + 8 more pages

Parke-Taylor amplitude

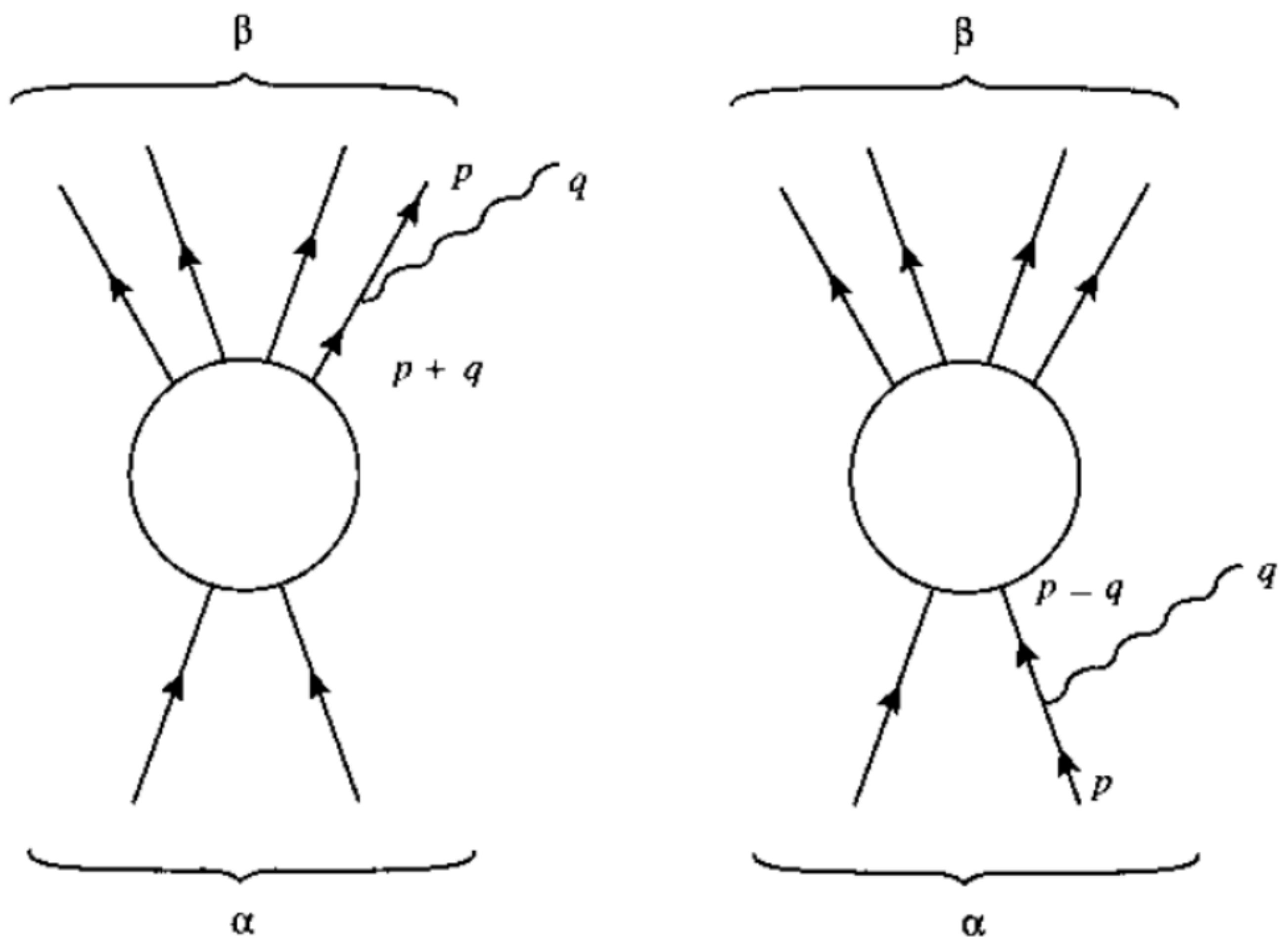
$$\mathcal{A}_n = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

Scattering Amplitudes

Universal relations!

Soft photon theorem

$$\lim_{q\to 0}\mathcal{A}_{n+1}=\sum_{i=1}^{n}Q_i\frac{\epsilon\cdot p_i}{q\cdot p_i}\mathcal{A}_n$$



Weinberg '65

Gauge invariance implies charge conservation

$$\mathcal{A}_{n+1}|_{\epsilon\to q}=0\Rightarrow\sum_{i=1}^{n}Q_i=0$$

Scattering Amplitudes

Hidden structure!

Gauge theory

$$\mathcal{A} = \sum_{i \in \Gamma} \frac{c_i n_i}{d_i}$$

$$c_i + c_j = c_k$$

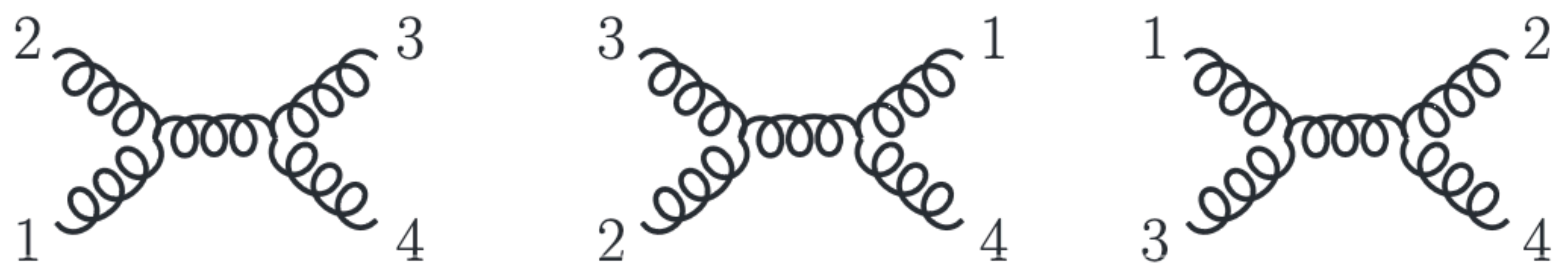


Figure 3: The three Feynman diagrams corresponding to the s, t and u channels.

Bern et.al. '19

Scattering Amplitudes

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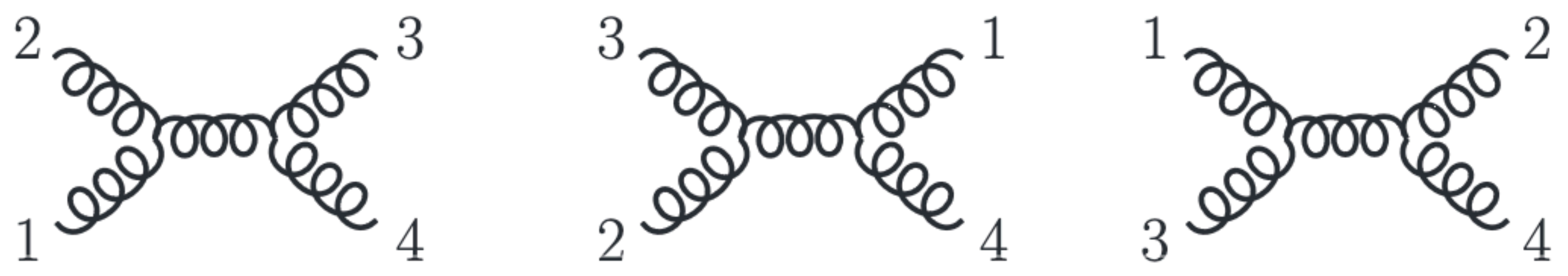
Color-kinematics duality

Bern, Carrasco, Johansson '08

$$c_i + c_j = c_k$$

$$n_i + n_j = n_k$$



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Gravity

$$\mathcal{M} = \sum_{i \in \Gamma} \frac{n_i n_i}{d_i}$$

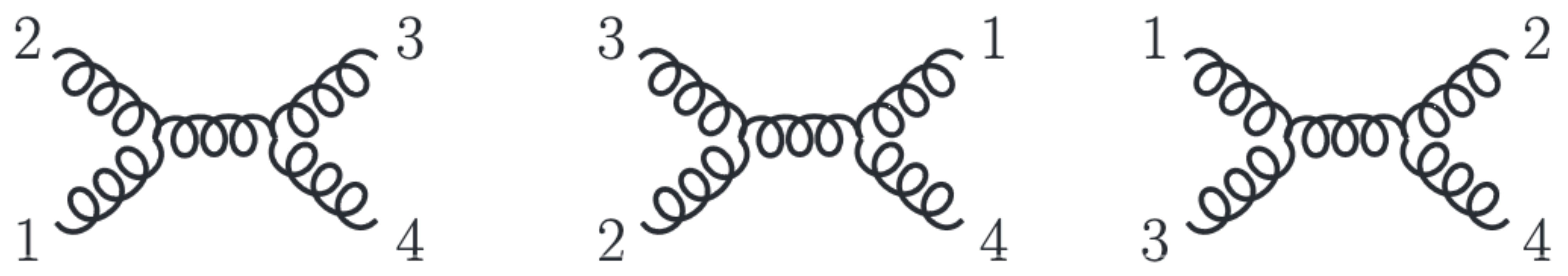

Figure 3: The three Feynman diagrams corresponding to the s, t and u channels.

Bern et.al. '19

Scattering Amplitudes

Surprising simplicity!

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi^i\partial^\mu\phi^i) + \lambda_{3,ijk}\phi^k(\partial_\mu\phi^i\partial^\mu\phi^j) + \lambda_{4,ijkl}\phi^k\phi^l(\partial_\mu\phi^i\partial^\mu\phi^j) + \ldots \\ &= \frac{1}{2}g_{ij}(\phi)(\partial_\mu\phi^i\partial^\mu\phi^j)\end{aligned}$$

Honerkamp, Meetz, … '70s

Scattering Amplitudes

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$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i \partial^\mu \phi^i) + \lambda_{3,ijk}\phi^k(\partial_\mu \phi^i \partial^\mu \phi^j) + \lambda_{4,ijkl}\phi^k \phi^l(\partial_\mu \phi^i \partial^\mu \phi^j) + \dots$$
$$= \frac{1}{2}g_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j)$$

Field redefinition $\phi^i \to \varphi^i(\phi)$

Honerkamp, Meetz, ... '70s

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$$= \frac{1}{2} g_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j)$$

Metric

Field redefinition $\phi^i \to \varphi^i(\phi)$

Christoffel symbol $\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{lk,j} + g_{jl,k} - g_{jk,l})$

Riemann curvature $R^i{}_{jkl} = \Gamma^i_{lj,k} + \Gamma^i_{kn}\Gamma^n_{lj} - (k \leftrightarrow l)$

Honerkamp, Meetz, ... '70s

Scattering Amplitudes

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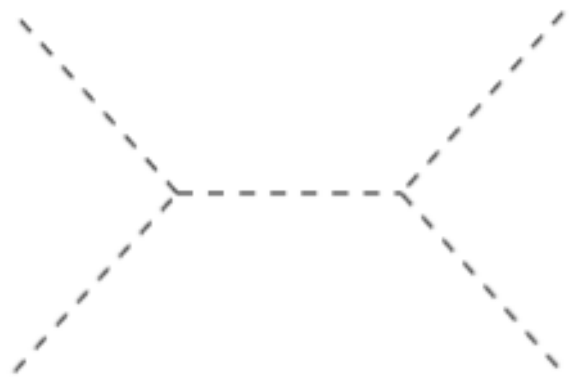
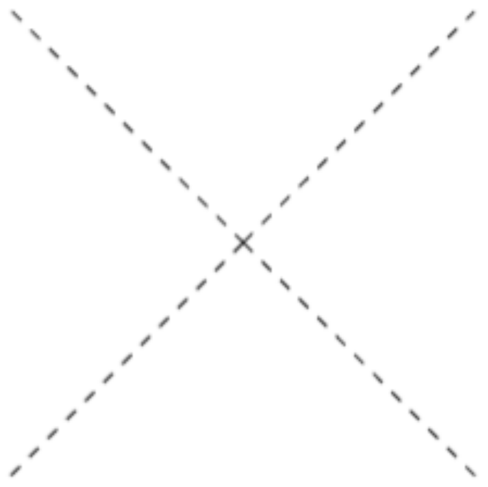
Feynman diagrams





$$\mathcal{A}_4 = (\partial g)(p \cdot p')\frac{1}{s_{ij}}(\partial g)(p \cdot p') + \cdots + (\partial^2 g)(p_i \cdot p_j)$$

Scattering Amplitudes

Surprising simplicity!

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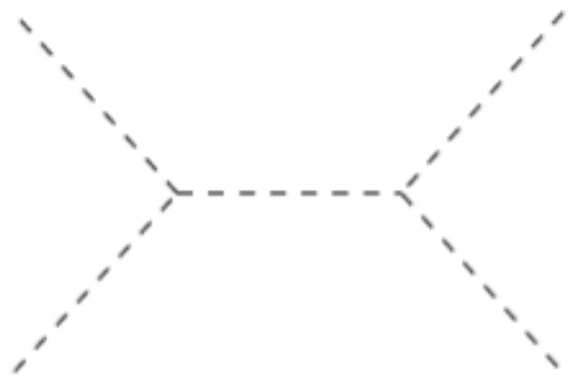
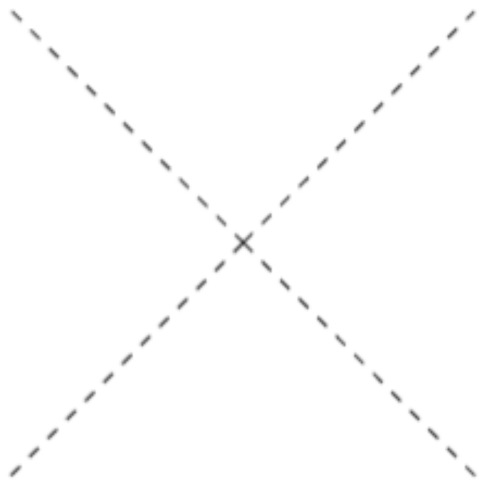
Feynman diagrams





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Geometric amplitude

$$\mathcal{A}_4 = R_{ikjl}s_{34} + R_{ijkl}s_{24}$$

$$s_{ij} = (p_i + p_j)^2$$

Scattering Amplitudes

Surprising simplicity!

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi^i \partial^\mu \phi^i) + \lambda_{3,ijk}\phi^k(\partial_\mu \phi^i \partial^\mu \phi^j) + \lambda_{4,ijkl}\phi^k\phi^l(\partial_\mu \phi^i \partial^\mu \phi^j) + \ldots \\ &= \frac{1}{2}g_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j)\end{aligned}$$

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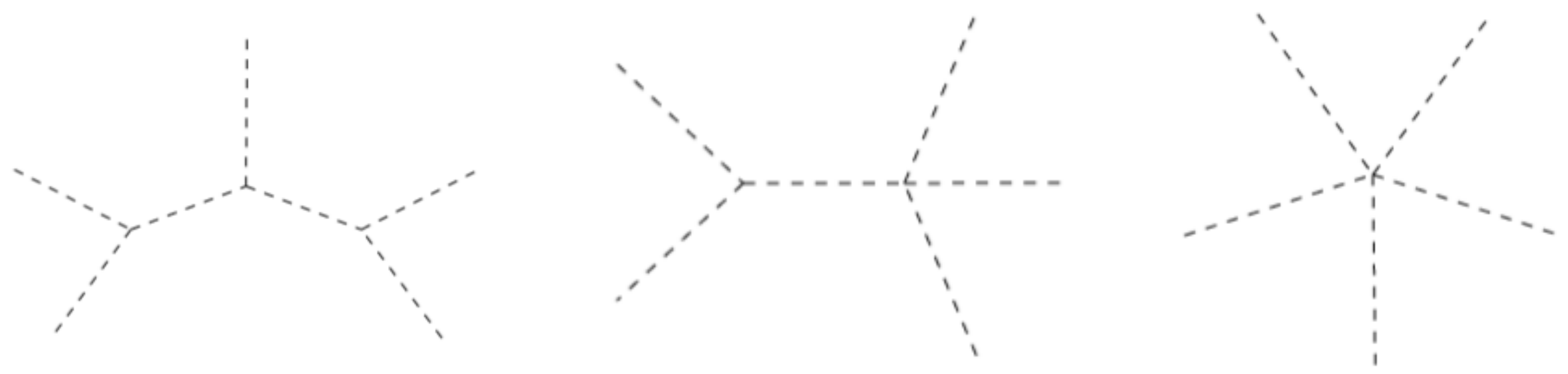

Geometric amplitude

$$\begin{aligned}\mathcal{A}_5 =& \nabla_k R_{iljm} s_{45} + \nabla_l R_{ikjm} s_{35} + \nabla_l R_{ijkm} s_{25} \\ &+ \nabla_m R_{ikjl} s_{34} + \nabla_m R_{ijkl}(s_{24} + s_{45})\end{aligned}$$

Scattering Amplitudes

Universal relations!

Geometric soft theorem

$$\lim_{q\to 0} \mathcal{A}_{n+1} = \nabla_i \mathcal{A}_n$$

$$\begin{aligned}\mathcal{A}_5 =&\nabla_k R_{iljm} s_{45} + \nabla_l R_{ikjm} s_{35} + \nabla_l R_{ijkm} s_{25} \\ &+ \nabla_m R_{ikjl} s_{34} + \nabla_m R_{ijkl}(s_{24} + s_{45})\end{aligned}$$

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Cheung, AH, Parra-Martinez ‘21

Scattering Amplitudes

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Cheung, AH, Parra-Martinez '21

Scattering Amplitudes

Hidden structure!

Nonlinear sigma model

$$\mathcal{A} = \sum_{i \in \Gamma} \frac{c_i n_i}{d_i}$$

$$c_i + c_j = c_k$$

FIG. 1. Trivalent graph for four particles.

$$c_i = R_{abcd} \sim f_{abx} f^x{}_{cd}$$

Scattering Amplitudes

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$$\mathcal{A} = \sum_{i \in \Gamma} \frac{c_i n_i}{d_i}$$

FIG. 1. Trivalent graph for four particles.

$$c_i = R_{abcd} \sim f_{abx} f^{x}{}_{cd}$$

$$n_i = s_{12}(s_{12} + 2s_{23})$$

Curvature-kinematics duality

$$c_i + c_j = c_k$$

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Galileon Theory

$$\mathcal{M} = \sum_{i \in \Gamma} \frac{n_i n_i}{d_i}$$

Double copy: Galileon theory = (NLSM)^2

Scattering Amplitudes

Hidden structure!

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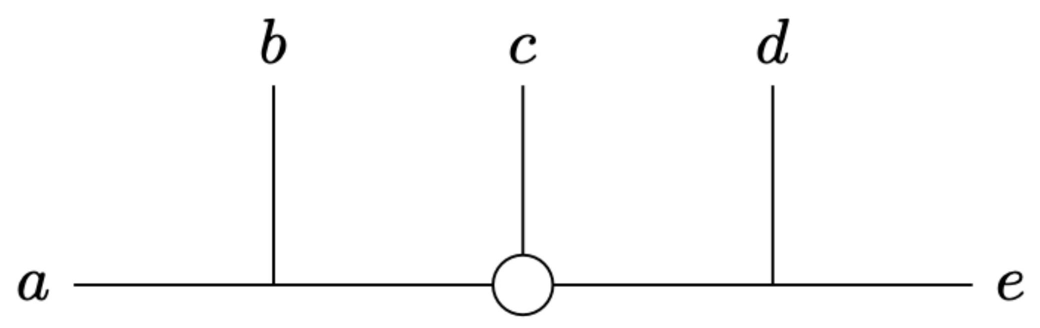

FIG. 2. Trivalent graph for five particles.

$$c_i = \nabla_c R_{abde} \neq f_{abx} f^{x}{}_{cy} f^{y}{}_{de}$$

Curvature-kinematics duality

$$c_i + c_j = c_k$$

$$n_i + n_j = n_k$$

Galileon Theory

$$\mathcal{M} = \sum_{i \in \Gamma} \frac{n_i n_i}{d_i}$$

Double copy: Galileon theory = (NLSM)^2

AH '24

Scattering Amplitudes

Geometry with fermions

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j) + i k_{\bar{p}r}(\phi) \bar{\psi}^{\bar{p}} \overleftrightarrow{\partial}_\mu \gamma^\mu \psi^r + i \omega_{\bar{p}ri}(\phi)(\partial_\mu \phi^i) \bar{\psi}^{\bar{p}} \gamma^\mu \psi^r$$

Derda, AH, Parra-Martinez '24

Scattering Amplitudes

Geometry with fermions

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j) + i k_{\bar{p}r}(\phi) \bar{\psi}^{\bar{p}} \overleftrightarrow{\partial}_\mu \gamma^\mu \psi^r + i \omega_{\bar{p}ri}(\phi)(\partial_\mu \phi^i) \bar{\psi}^{\bar{p}} \gamma^\mu \psi^r$$

Surprising structure!

$$\mathcal{A}_4 = (\bar{u} \not{p}_j u) \bar{R}_{\bar{p}rij}$$

$$\mathcal{A}_5 = (\bar{u} \not{p}_j u) \nabla_k \bar{R}_{\bar{p}rij} + (\bar{u} \not{p}_k u) \nabla_j \bar{R}_{\bar{p}rik}$$

Derda, AH, Parra-Martinez ‘24

Scattering Amplitudes

Geometry with fermions

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)(\partial_\mu \phi^i \partial^\mu \phi^j) + i k_{\bar{p}r}(\phi) \bar{\psi}^{\bar{p}} \overleftrightarrow{\partial}_\mu \gamma^\mu \psi^r + i \omega_{\bar{p}ri}(\phi)(\partial_\mu \phi^i) \bar{\psi}^{\bar{p}} \gamma^\mu \psi^r$$

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Geometric soft theorem

$$\lim_{q \to 0} \mathcal{A}_{n+1} = \nabla_i \mathcal{A}_n$$

Derda, AH, Parra-Martinez '24

Standard Model Effective Field Theory

Encode heavy new physics in effective operators

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{C_i}{\Lambda^2}\mathcal{O}_i + \ldots$$

X^3

Q_G	${}^6Q_{G^3}$	$f^{\mathscr{A}\mathscr{B}\mathscr{C}} G_\mu^{\mathscr{A}\nu} G_\nu^{\mathscr{B}\rho} G_\rho^{\mathscr{C}\mu}$
Q_W	${}^6Q_{W^3}$	$\epsilon^{abc} W_\mu^{a\,\nu} W_\nu^{b\,\rho} W_\rho^{c\,\mu}$

H^6

Q_H	${}^6Q_{H^6}$	$(H^\dagger H)^3$

H^4D^2

$Q_{H\Box}$	${}^6Q_{H^4\Box}$	$(H^\dagger H)\Box(H^\dagger H)$
Q_{HD}	${}^6Q_{H^4D^2}$	$(D^\mu H^\dagger H)\,(H^\dagger D_\mu H)$

X^2H^2

Q_{HG}	${}^6Q_{G^2H^2}$	$(H^\dagger H)\, G^{\mathscr{A}}_{\mu\nu} G^{\mathscr{A}\mu\nu}$
Q_{HW}	${}^6Q_{W^2H^2}$	$(H^\dagger H)\, W^a_{\mu\nu} W^{a\mu\nu}$
Q_{HB}	${}^6Q_{B^2H^2}$	$(H^\dagger H)\, B_{\mu\nu} B^{\mu\nu}$
Q_{HWB}	${}^6Q_{WBH^2}$	$(H^\dagger \tau^a H)\, W^a_{\mu\nu} B^{\mu\nu}$

Table 3: Bosonic even-parity dimension-six operators in the SMEFT. The first column is the notation of Ref. [26], and the second column is the notation used in this paper.

Standard Model Effective Field Theory

Renormalization Group Equations

$$\mathcal{L}_{\rm SMEFT} = \mathcal{L}_{\rm SM} + \sum_i \frac{C_i}{\Lambda^2}\mathcal{O}_i + \dots$$

1-loop UV divergence

$$\Delta S = \frac{1}{32\pi\epsilon}\int d^4x \left\{\frac{1}{12}\mathrm{Tr}[Y_{\mu\nu}Y^{\mu\nu}] + \frac{1}{2}\mathrm{Tr}[X^2]\right\}$$

Curvature shows up

$$Y_{\mu\nu} = R^i{}_{jkl}(D_\mu Z)^k(D_\nu Z)^l + F^a_{\mu\nu}\nabla_j t^i_a$$

AH, Jenkins, Manohar ‘22

Assi, AH, Manohar, Pages, Shen ‘23

Standard Model Effective Field Theory

Renormalization Group Equations

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{C_i}{\Lambda^2} \mathcal{O}_i + \dots$$

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Curvature shows up

$$Y_{\mu\nu} = R^i{}_{jkl} (D_\mu Z)^k (D_\nu Z)^l + F^a_{\mu\nu} \nabla_j t^i_a$$

SMEFT RGE to dimension 8

$$\begin{aligned}
{}^8\dot{C}^{(1)}_{H^6D^2} = &- 96 \ {}^6C_{H^6} {}^6C_{H^4\Box} - 12 \ {}^6C_{H^6} {}^6C_{H^4D^2} + \left(352\lambda + 20g_1^2 + \frac{20}{3} g_2^2\right) \left({}^6C_{H^4\Box}\right)^2 \\
&+ \left(-23\lambda + \frac{1}{8} g_1^2 + \frac{161}{24} g_2^2\right) \left({}^6C_{H^4D^2}\right)^2 + \left(-64\lambda - 2g_1^2 + 12g_2^2\right) {}^6C_{H^4\Box} {}^6C_{H^4D^2} \\
&- 22g_2^2 \ {}^6C_{H^4\Box} {}^6C_{W^2H^2} + 6g_1^2 \ {}^6C_{H^4\Box} {}^6C_{B^2H^2} - \frac{32}{3} g_1 g_2 \ {}^6C_{H^4\Box} {}^6C_{WBH^2} \\
&+ 8g_2^2 \ {}^6C_{H^4D^2} {}^6C_{W^2H^2} + 6g_1^2 \ {}^6C_{H^4D^2} {}^6C_{B^2H^2} + \frac{43}{3} g_1 g_2 \ {}^6C_{H^4D^2} {}^6C_{WBH^2} \\
&+ 512\lambda \left({}^6C_{G^2H^2}\right)^2 + \left(192\lambda + 4g_2^2\right) \left({}^6C_{W^2H^2}\right)^2 + \left(64\lambda + 12g_1^2\right) \left({}^6C_{B^2H^2}\right)^2 \\
&+ \left(-3g_1^2 - 3g_2^2\right) \left({}^6C_{WBH^2}\right)^2 + \frac{80}{3} g_1 g_2 \ {}^6C_{W^2H^2} {}^6C_{WBH^2} + \frac{8}{3} g_1 g_2 \ {}^6C_{B^2H^2} {}^6C_{WBH^2} \\
&+ \left(68\lambda + \frac{1}{2} g_1^2 - \frac{31}{6} g_2^2\right) {}^8C^{(1)}_{H^6D^2} + \left(-8\lambda + 7g_1^2 + \frac{17}{3} g_2^2\right) {}^8C^{(2)}_{H^6D^2} \,, \qquad \text{(C.23)}
\end{aligned}$$

AH, Jenkins, Manohar '22

Assi, AH, Manohar, Pages, Shen '23

Scattering Amplitudes

Surprising simplicity! $\mathcal{A}_4 = R_{ikjl}s_{34} + R_{ijkl}s_{24}$

Universal relations! $\lim_{q\to 0} \mathcal{A}_{n+1} = \nabla_i \mathcal{A}_n$

Hidden structure! Double copy: Galileon theory = (NLSM)^2

Practical calculations RGE for Standard Model Effective Field Theory

Thank you!