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Preserving continuum space-time symmetries on the lattice

Alexander Rothkopf
Faculty of Science and Technology
Department of Mathematics and Physics
University of Stavanger

in collaboration with Jan Nordström (LiU) & Will Horowitz (UCT)

A.R., W. Horowitz and J. Nordström: arXiv:2404.18676

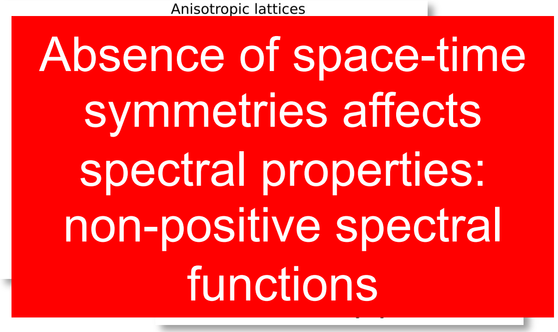
see also JCP 498 (2024) 112652

NPACT
Norwegian Particle, Astroparticle & Cosmology Theory network

The Broader Picture

Quantum Boundary Value Problems

Euclidean Lattice QCD

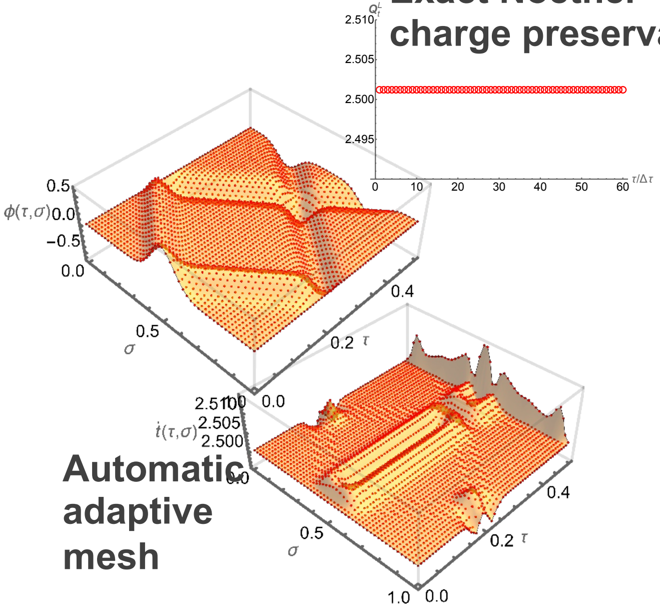


Absence of space-time symmetries affects spectral properties: non-positive spectral functions

R. Larsen, G. Parkar, A.R. J. Weber
arXiv:2402.10819

R. Larsen, A.R. & HotQCD
PRD 109 (2024) 7, 074504

Exact Noether charge preservation



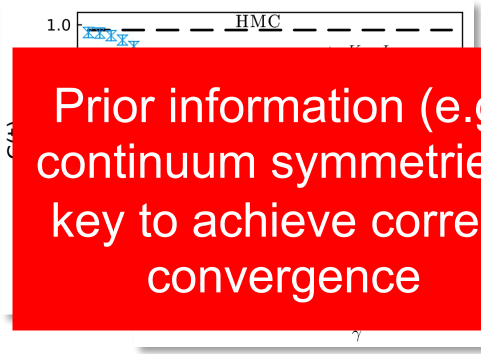
Automatic adaptive mesh

Novel SCL action

Classical Initial Value Problems

Quantum Initial Value Problems

Complex Langevin



Prior information (e.g. continuum symmetries) key to achieve correct convergence

D. Alvestad, A.R., D. Sexty
PRD 109 (2024) 3, L031502

Outline

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- Motivation – The Broader Picture
- From the world-line formalism to a new action for classical fields (SCL)
- The discretized classical initial-boundary value problem
- Scalar wave propagation in (1+1)d as proof-of-principle
- Summary

Worldline Formalism in GR

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- Relativistic point particle motion: "shortest path in given space-time" = geodesic
- Equal treatment of space & time as **dynamic coordinate maps**: from trajectory to world line [both t(ɣ) and x(ɣ) evolve dynamically]

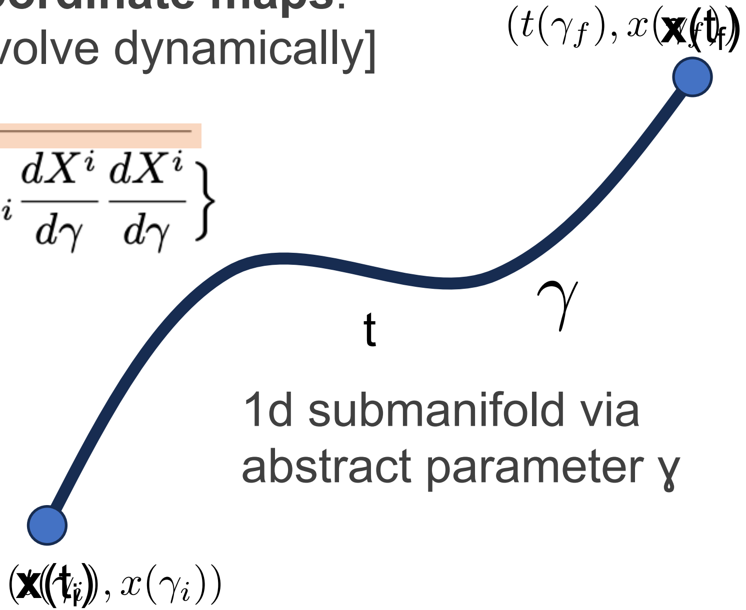
$$S_{\rm geo} = \int d\gamma\,(-mc)\Big\{\sqrt{\Big(G_{00} + \frac{V(\vec{x})}{2mc^2}\Big)\frac{dX^0}{d\gamma}\frac{dX^0}{d\gamma} + G_{ii}\frac{dX^i}{d\gamma}\frac{dX^i}{d\gamma}}\Big\}$$

$$V(\vec{x})/2mc^2 \ll 1$$

$$\frac{d|\vec{x}|/d\gamma}{dt/d\gamma}/c = v/c \ll 1$$

$$X^\mu = (t, \vec{x})^\mu$$

$$S_{\rm nr} = \int dt\,\Big\{-mc^2 + \frac{1}{2}m\dot{\vec{x}}^2(t) - V\big(\vec{x}(t)\big)\Big\}$$



- mc denotes scale where motion through space and time becomes inseparable

Advantages of the worldline formalism

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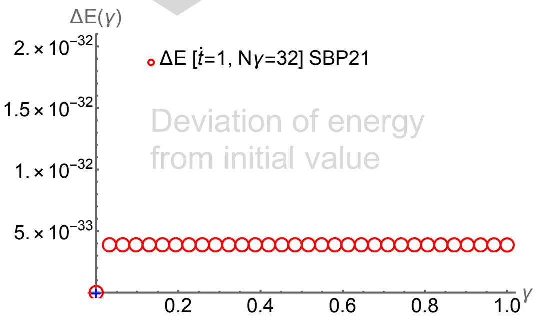
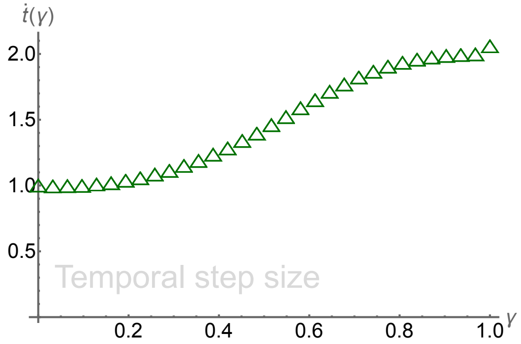
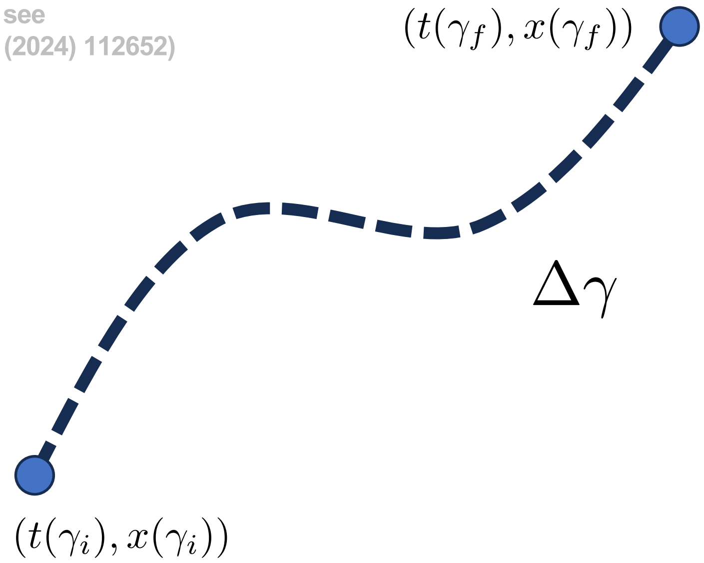
- Discretizing the action in γ leaves space-time coordinates $X^\mu = (t, \vec{x})^\mu$ continuous
- Discretized world-line action invariant under infinitesimal coordinate transforms: Noether's theorem holds! (for a detailed study of point mechanics see A.R., J. Nordström, J.Comput.Phys. 498 (2024) 112652)



Energy of the system preserved *exactly* at its *continuum* value

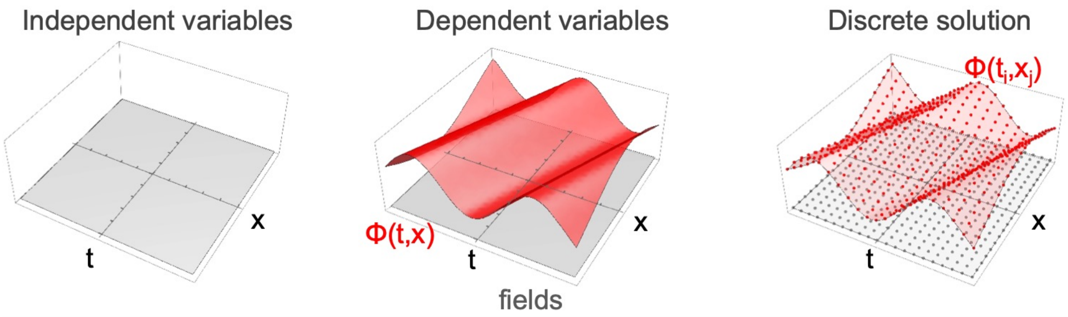


Resolution of the time grid *adapts* to dynamics of particle



A Field Theory Counterpart?

Conventional field theory



Spacetime symmetries broken by Δt and Δx

Field theory with dynamic coordinate maps

Spacetime symmetries unaffected by Δτ and Δσ?

A world ”volume” action for fields?

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- Starting point is the standard reparameterization invariant action

$$S = \int d^{(d+1)}X \sqrt{-\det[G]}\frac{1}{2}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) - V(\phi)\Big)$$

- Are we perhaps overlooking a constant term, just as in the non-relativistic action?

$$S = \int d^{(d+1)}X \sqrt{-\det[G]}\Big\{-T + \frac{1}{2}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) + V(\phi)\Big)\Big\}.$$

- Consider as low energy limit of another more general action (κ = energy density/T)

$$\mathcal{S}_{\rm BVP} \equiv \int d^{(d+1)}X \sqrt{-\det[G]}(-T)\Big\{1 - \frac{1}{2T}\Big(G^{\mu\nu}\partial_\mu\phi(X)\partial_\nu\phi(X) + V(\phi)\Big)\Big\} + \mathcal{O}(\kappa^2)$$

Towards the SCL action

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- Crucial next step: elevate spacetime coordinates to dynamical coordinate maps

worldline: $t \to t(\gamma)$ here: $X^\mu \to X^\mu(\Sigma)$ $\qquad \Sigma^a = (\tau, \vec{\sigma})^a = (\tau, \sigma_1, \ldots, \sigma_d)^a$

$$\mathcal{S}_{\rm BVP} = \int d^{(d+1)}\Sigma \, (-T)\sqrt{\Big(\frac{1}{T}V(\phi) - 1\Big)\det[g] + \frac{1}{T}\partial_a\phi(\Sigma)\partial_b\phi(\Sigma)\mathrm{adj}[g]_{ab}}.$$

$$\mathrm{adj}[g] = g^{-1}\det[g]$$

- Can absorb the Jacobian into new *induced metric g* on the space of parameters

$$\sqrt{-\det[J]\det[G]\det[J]} = \sqrt{-\det[J^T]\det[G]\det[J]} = \sqrt{-\det[J^T G J]} = \sqrt{-\det[g]}$$

- The scale T denotes where field and coordinate dynamics become inseparable

Next steps

$$\mathcal{S}_{\rm BVP} = \int d^{(d+1)}\Sigma \,(-T)\sqrt{\Big(\frac{1}{T}V(\phi) - 1\Big)\det[g] + \frac{1}{T}\partial_a\phi(\Sigma)\partial_b\phi(\Sigma)\mathrm{adj}[g]_{ab}}.$$

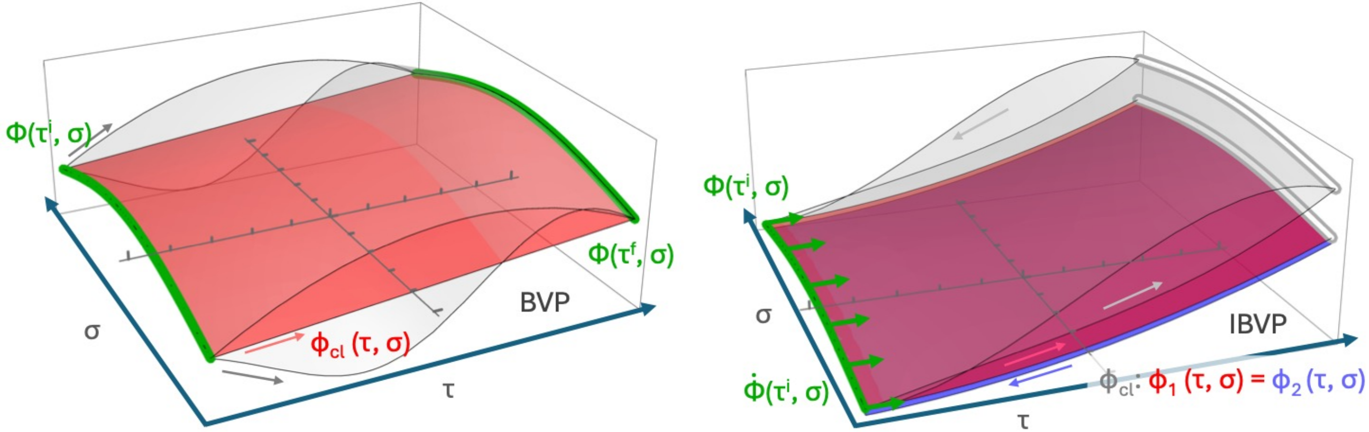
- Formulate an initial value problem version of the SCL action
- Discretize the action and show that Noether’s theorem holds for Poincare group
- Demonstrate numerically feasibility of locating critical point of the action: classical field solution without the need to solve Euler-Lagrange equations

$$\mathcal{E}_{\rm BVP} = \int d^{(d+1)}\Sigma\; E_{\rm BVP} = \int d^{(d+1)}\Sigma\,\frac{1}{2}\Big\{\Big(\frac{1}{T}V(\phi) - 1\Big)\det[g] + \frac{1}{T}\partial_a\phi(\Sigma)\partial_b\phi(\Sigma)\mathrm{adj}[g]_{ab}\Big\}$$

Classical Schwinger Keldysh (Galley)

- Doubling of all degrees of freedom by introducing forward and backward branches

see C. Galley PRL 110 (2013) 17, 174301 and A.R., J. Nordström JCP 477 (2023) 111942



$$\mathcal{E}_{\rm IBVP} = \int d^{(d+1)}\Sigma \left\{ E_{\rm BVP}[X_1, \partial_a X_1, \phi_1, \partial_a \phi_1] - E_{\rm BVP}[X_2, \partial_a X_2, \phi_2, \partial_a \phi_2] \right\}$$

connecting conditions

$$X_1^\mu(\tau = \tau^{\rm f}, \vec{\sigma}) = X_2^\mu(\tau = \tau^{\rm f}, \vec{\sigma}), \qquad \phi_1(\tau = \tau^{\rm f}, \vec{\sigma}) = \phi_2(\tau = \tau^{\rm f}, \vec{\sigma})$$

$$\partial_0 X_1^\mu|_{\tau=\tau^{\rm f}} = \partial_0 X_2^\mu|_{\tau=\tau^{\rm f}}, \qquad \partial_0 \phi_1|_{\tau=\tau^{\rm f}} = \partial_0 \phi_2|_{\tau=\tau^{\rm f}}.$$

Summation-by-parts finite differences

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- Derivation of Noether theorem or governing equations rely on integration by parts
- Mimetic discretization needed to preserve IBP in discrete setting:

for a review see D. Fernández, J. E. Hicken, and D. W. Zingg, Comp. & Fluids 95 171 (2014)

$$\int_{t_i}^{t_f} dt\, u(t)\, v(t) \approx \mathbf{u}^t\, \mathbb{H}\, \mathbf{v}$$

quadrature rule

$$\mathbb{D} = \mathbb{H}^{-1}\mathbb{Q}$$

finite difference stencil

$$\mathbb{Q} + \mathbb{Q}^t = \mathbb{E}_N - \mathbb{E}_0$$

$$= \mathrm{diag}[-1, 0, \ldots, 0, 1]$$

$$\Delta t \begin{bmatrix} \frac{1}{2} & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{2} \end{bmatrix}$$

$\mathbb{H}^{[2,1]}$

$$\frac{1}{\Delta t}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$\mathbb{D}^{[2,1]}$

$$(\mathbb{D}\mathbf{u})^t\, \mathbb{H}\, \mathbf{v} = -\mathbf{u}^t\, \mathbb{H}\, \mathbb{D}\mathbf{v} + \mathbf{u}_N \mathbf{v}_N - \mathbf{u}_0 \mathbf{v}_0$$

Avoiding the doubler problem

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- Symmetric stencil leads to appearance of doubler modes when naïve SBP is used
- Wilson term trick not applicable: derivative acts on real-valued functions
- Modern approach in PDE community: weakly enforced boundary data

A. Rothkopf, J. Nordström, JCP 477 (2023) 111942

Affine coordinate formulation

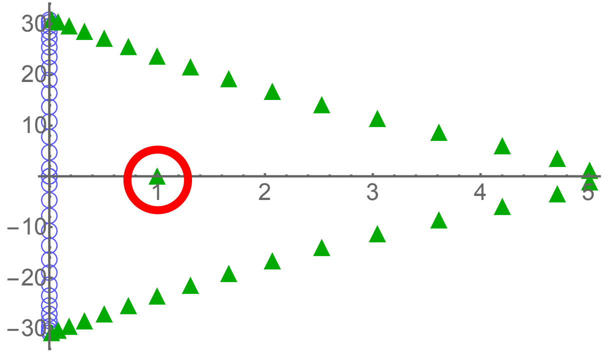
$$S = \int dt\,(\dot{x}(t)\dot{x}(t)) \quad x(0) = x_i$$

$$S \approx (\mathbb{D}\mathbf{x})^t \mathbb{H}\mathbb{D}\mathbf{x}$$

$$\bar{\mathbb{D}}\mathbf{x} = \mathbb{D}\mathbf{x} + \mathbb{H}^{-1}\mathbb{E}_0(\mathbf{x} - \mathbf{x}_i)$$

Modification acting on the path x itself

constant shift



+ all zero modes are lifted

+ physical constant mode with correct IC as unit EV

The discretized action

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- Due to mimetic nature of SBP operator simply replace derivatives by $\mathbb{D}$

$$\mathbb{E}_{\rm BVP}[\boldsymbol{X}_1^\mu, \bar{\mathbb{D}}_a^\mu \boldsymbol{X}_1^\mu, \boldsymbol{\phi}_1, \bar{\mathbb{D}}_a^\phi \boldsymbol{\phi}_1] =$$
$$\frac{1}{2}\Big\{\Big(\frac{1}{T}V(\boldsymbol{\phi}_1) - 1\Big) \circ \det[\boldsymbol{g}_1] + \frac{1}{T}(\bar{\mathbb{D}}_a^\phi \boldsymbol{\phi}_1) \circ (\bar{\mathbb{D}}_b^\phi \boldsymbol{\phi}_1) \circ \mathrm{adj}[\boldsymbol{g}_1]_{ab}\Big\}^T \boldsymbol{h}$$

$$\boldsymbol{g}_{ab} = G_{\mu\nu}(\bar{\mathbb{D}}_a^\mu \boldsymbol{X}^\mu) \circ (\bar{\mathbb{D}}_b^\nu \boldsymbol{X}^\nu), \quad \det[\boldsymbol{g}] = \sum_{i_0,\ldots,i_d} \epsilon_{i_0\cdots i_d} \boldsymbol{g}_{0,i_0} \circ \cdots \circ \boldsymbol{g}_{d,i_d}$$

- **Discrete action** remains manifest **invariant under Poincare transformations**
- Integration by parts exactly mimicked: Noether current & charge as in continuum

$$\boldsymbol{q}^{\rm L} = \frac{\partial \mathbb{E}^{\rm L}_{\rm BVP}}{\partial(\mathbb{D}_0 \mathbf{X}^\mu)} \delta \mathbf{X}^\mu = \Big(\frac{\partial \mathbb{E}_{\rm BVP}}{\partial(\mathbb{D}_0 \mathbf{X}^\mu)} + \tilde{\boldsymbol{\lambda}}_\mu \circ \mathfrak{d}^0[0] + \tilde{\boldsymbol{\gamma}}_\mu \circ \mathfrak{d}^0[N_0]\Big)\delta \mathbf{X}^\mu,$$

$$\boldsymbol{Q}^{\rm L} = \Big(\mathbb{H}_\sigma \frac{\partial \mathbb{E}_{\rm BVP}}{\partial(\mathbb{D}_0 \mathbf{X}^\mu)} + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\lambda}}_\mu)\mathfrak{d}^0[0] + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\gamma}}_\mu)\mathfrak{d}^0[N_0]\Big)\delta \mathbf{X}^\mu.$$

Proof-of-principle in (1+1)d

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- Scalar wave propagation is numerically challenging (stability, accuracy)

$$\mathcal{S}_{\rm BVP} = \int d\tau d\sigma \, (-T)\sqrt{\; - \det[g] + \frac{1}{T}\partial_a \phi(\Sigma)\partial_b\phi(\Sigma){\rm adj}[g]_{ab}}$$

$$= \int d\tau d\sigma \, (-T)\Big\{ c^2(\dot{t}x' - \dot{x}t')^2 + \frac{1}{T}\Big(\dot{\phi}^2(c^2(t')^2 - (x')^2) + 2\dot{\phi}\phi'(\dot{x}x' - c^2\dot{t}t') + (\phi')^2(c^2\dot{t}^2 - \dot{x}^2)\Big)\Big\}^{1/2}$$

- Simplify by considering only time as dynamical mapping (trivial x[τ,σ]= σ)

$$\mathcal{E}_{\rm BVP} \overset{x=\sigma}{\equiv} \int d\tau d\sigma \, \frac{1}{2}\Big\{(\dot{t})^2 + \frac{1}{T}\Big(\dot{\phi}^2((t')^2 - 1) - 2\dot{\phi}\phi'\dot{t}t' + (\phi')^2(\dot{t}^2)\Big)\Big\}$$

Discretized IBVP action

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- Introduce forward and backward branch (classical Schwinger-Keldysh)

$$\begin{aligned}\mathbb{E}^{\rm L}_{\rm IBVP} =& \frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_1)^2 - 1\big)\\ &- 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1)\circ(\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_1)\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)\circ(\mathbb{D}^t_\sigma \boldsymbol{t}_1) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_1)^2\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_1)^2\Big)\Big\}^T \boldsymbol{h}\\ &-\frac{1}{2}\Big\{(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2 + \frac{1}{T}\Big((\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2)^2 \circ \big((\mathbb{D}_\sigma \boldsymbol{t}_2)^2 - 1\big)\\ &- 2(\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2)\circ(\bar{\mathbb{D}}^\phi_\tau \boldsymbol{\phi}_2)\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)\circ(\mathbb{D}^t_\sigma \boldsymbol{t}_2) + (\bar{\mathbb{D}}^\phi_\sigma \boldsymbol{\phi}_2)^2\circ(\bar{\mathbb{D}}^t_\tau \boldsymbol{t}_2)^2\Big)\Big\}^T \boldsymbol{h}\end{aligned}$$

- Enforce initial (temporal), Dirichlet boundary (spatial) and connecting conditions

$$\begin{aligned}&+(\boldsymbol{\lambda}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[\boldsymbol{t}_1] - \boldsymbol{t}_{\rm IC}\big) + (\boldsymbol{\lambda}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[\boldsymbol{\phi}_1] - \boldsymbol{\phi}_{\rm IC}\big)\\ &+(\tilde{\boldsymbol{\lambda}}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[(\mathbb{D}_\tau \boldsymbol{t}_1)] - \dot{\boldsymbol{t}}_{\rm IC}\big) + (\tilde{\boldsymbol{\lambda}}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^0_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_1)] - \dot{\boldsymbol{\phi}}_{\rm IC}\big)\\ &+(\boldsymbol{\gamma}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[\boldsymbol{t}_1] - \mathbb{P}^{N_\tau}_\tau[\boldsymbol{t}_2]\big) + (\boldsymbol{\gamma}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[\boldsymbol{\phi}_1] - \mathbb{P}^{N_\tau}_\tau[\boldsymbol{\phi}_2]\big)\end{aligned}$$

$$\begin{aligned}&+(\tilde{\boldsymbol{\gamma}}^t)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{t}_1)] - \mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{t}_2)]\big)\\ &+(\tilde{\boldsymbol{\gamma}}^\phi)^T \mathbb{h}_\sigma\big(\mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_1)] - \mathbb{P}^{N_\tau}_\tau[(\mathbb{D}_\tau \boldsymbol{\phi}_2)]\big)\\ &+(\boldsymbol{\kappa}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^0_\sigma[\boldsymbol{\phi}_1] - \boldsymbol{0}\big) + (\tilde{\boldsymbol{\kappa}}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^{N_\sigma}_\sigma[\boldsymbol{\phi}_1] - \boldsymbol{0}\big)\\ &+(\boldsymbol{\xi}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^0_\sigma[\boldsymbol{\phi}_2] - \boldsymbol{0}\big) + (\tilde{\boldsymbol{\xi}}^\phi)^T \mathbb{h}_\tau\big(\mathbb{P}^{N_\sigma}_\sigma[\boldsymbol{\phi}_2] - \boldsymbol{0}\big).\end{aligned}$$

- Locate extremum via numerical optimization (Interior Point Optimization)

Proof-of-principle in (1+1)d

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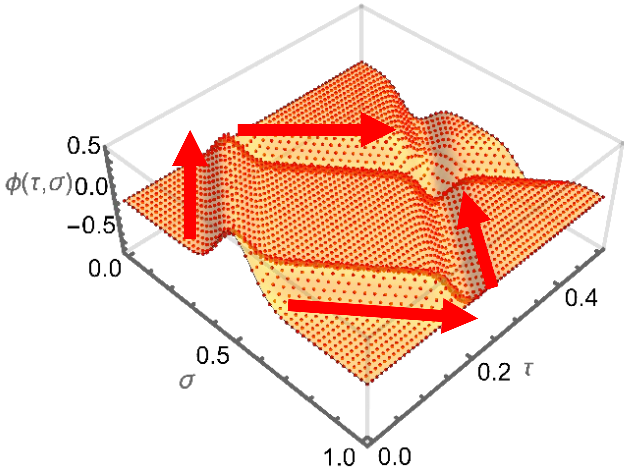
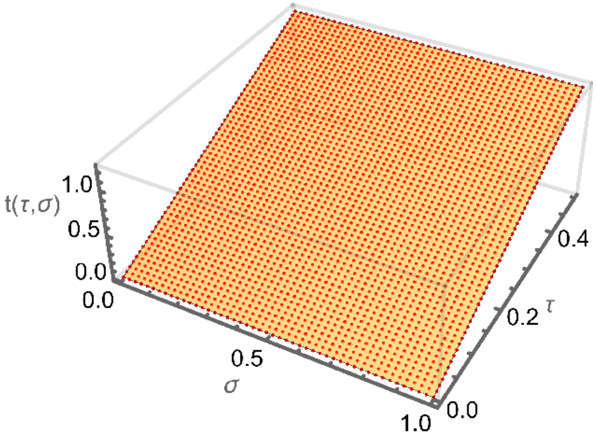
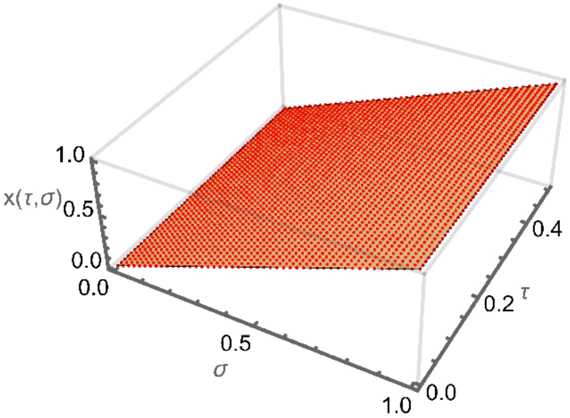
- Left- and right propagating wave-packages bouncing off a stiff wall



field evolution ϕ[τ,σ]



temporal map t[τ,σ]



trivial spatial map x[τ,σ]= σ

- Here T=10.000, choice to obtain effects on the coordinate maps on percent level

Automatic spacetime mesh refinement

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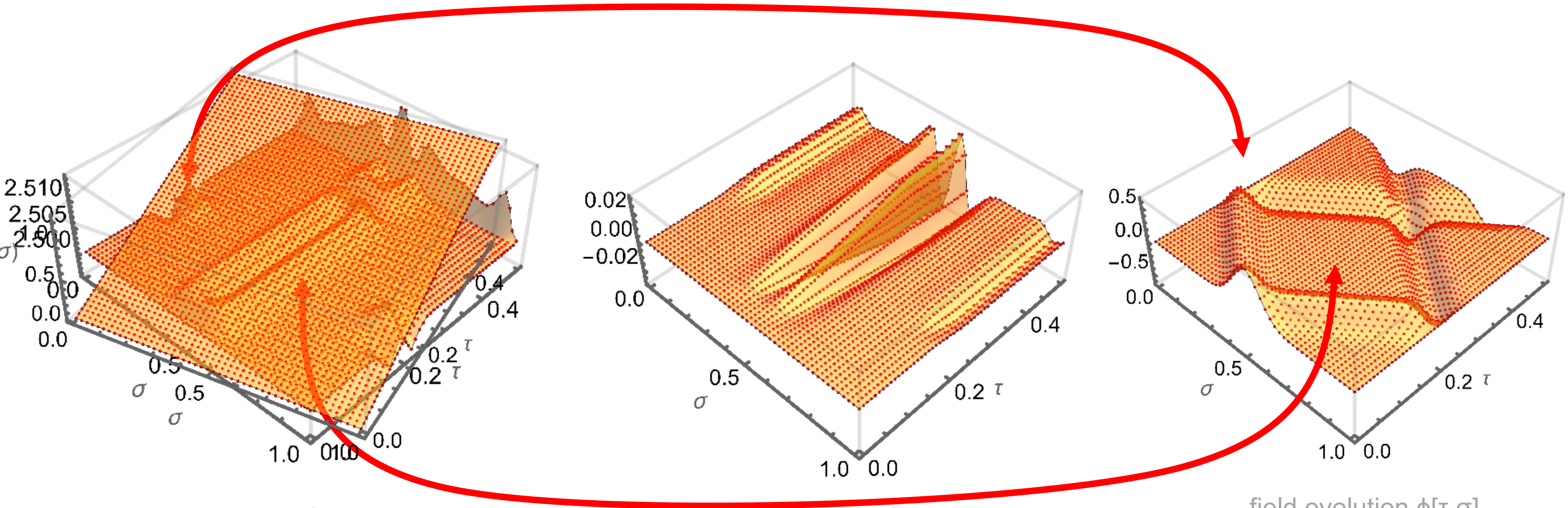


τ derivative of t[τ,σ]

σ derivative of t[τ,σ]

field evolution ϕ[τ,σ]

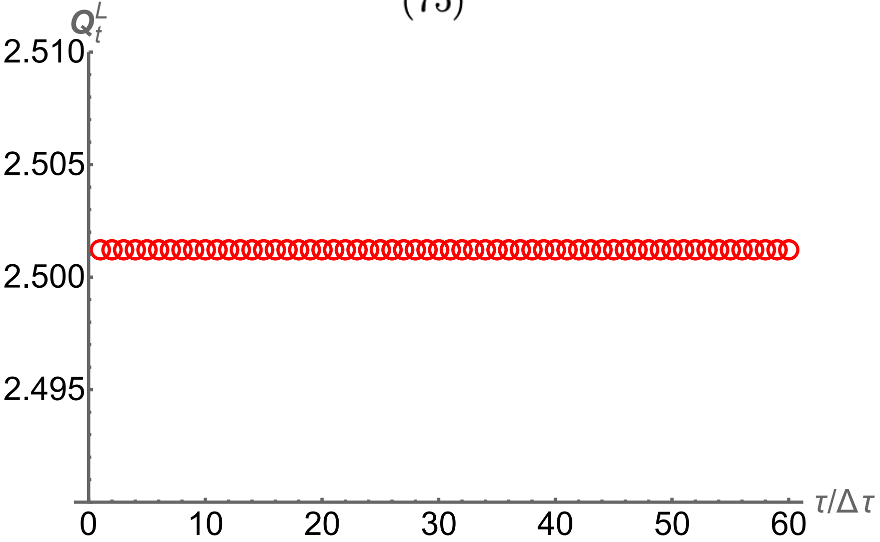
- Temporal map automatically adapts resolution according to wave dynamics

Noether Charge

- Due to mimetic SBP discretization: continuum expression with

$$\boldsymbol{Q}_t^{\mathrm{L}} = \mathbb{H}_\sigma \underbrace{\left\{(\mathbb{D}_\tau \boldsymbol{t}_1) + \frac{1}{T}\Big((\mathbb{D}_\sigma \boldsymbol{\phi}_1)^2 \circ (\mathbb{D}_\tau \boldsymbol{t}_1) - (\mathbb{D}_\tau \boldsymbol{\phi}_1) \circ (\mathbb{D}_\sigma \boldsymbol{\phi}_1) \circ (\mathbb{D}_\sigma \boldsymbol{t}_1)\Big)\right\}}_{\boldsymbol{J}^0 \in \mathbb{R}^{N_\tau \times N_\sigma}}$$
$$+ \underbrace{\left\{(\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\lambda}}^t)\boldsymbol{\mathfrak{d}}^\tau[0] + (\boldsymbol{h}_\sigma^T \tilde{\boldsymbol{\gamma}}^t)\boldsymbol{\mathfrak{d}}^\tau[N_\tau]\right\}}_{\text{Lagr. mult. contrib.}}, \qquad (75)$$

- Exact conservation of the Noether charge associated with time translations.



Summary

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- World-line formalism suggests dynamical coordinate maps are essential
- SCL action incorporates **dynamical coordinate maps** with field d.o.f.s
- Discretization via **summation-by-parts** mimetic finite difference scheme
- Discretization of abstract parameter action **retains space-time symmetries**
- **Dynamical** emergence of **time-mesh** & **exact conservation** of Noether charge

Inclusion of initial & boundary conditions

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- In contrast to implicit analytic treatment make explicit via Lagrange multipliers

see also A.R., J. Nordström JCP 511 (2024) 113138

$$\mathcal{E}^{\rm L}_{\rm IBVP} = \int d^{(d+1)}\Sigma \Big\{ E_{\rm BVP}[X_1,\partial_a X_1,\phi_1,\partial_a\phi_1] - E_{\rm BVP}[X_2,\partial_a X_2,\phi_2,\partial_a\phi_2]\Big\}$$
$$+\int\prod_{a=1}^{d} d\Sigma_a \Big\{\lambda_\mu\big(X_1^\mu(\tau^{\rm i},\vec\sigma)-X^\mu_{\rm IC}\big)+\lambda_\phi\big(\phi_1(\tau^{\rm i},\vec\sigma)-\phi_{\rm IC}\big)$$
$$+\tilde\lambda_\mu\big(\partial_0 X_1^\mu(\tau^{\rm i},\vec\sigma)-\dot X^\mu_{\rm IC}\big)+\tilde\lambda_\phi\big(\partial_0\phi_1(\tau^{\rm i},\vec\sigma)-\dot\phi_{\rm IC}\big)$$
$$+\gamma_\mu\big(X_1^\mu(\tau^{\rm f},\vec\sigma)-X_2^\mu(\tau^{\rm f},\vec\sigma)\big)+\gamma_\phi\big(\phi_1(\tau^{\rm f},\vec\sigma)-\phi_2(\tau^{\rm f},\vec\sigma)\big)$$
$$+\tilde\gamma_\mu\big(\partial_0 X_2^\mu(\tau^{\rm f},\vec\sigma)-\partial_0 X_2^\mu(\tau^{\rm f},\vec\sigma)\big)+\tilde\gamma_\phi\big(\partial_0\phi_1(\tau^{\rm f},\vec\sigma)-\partial_0\phi_2(\tau^{\rm f},\vec\sigma)\big)\Big\}$$
$$+\sum_{j=1}^{d}\int\prod_{\substack{a=0\\a\neq j}}^{d} d\Sigma_a\Big\{\kappa^j_\mu\big(X_1^\mu(\sigma_j^{\rm i})-X^\mu_{\rm sBCL}(\sigma_j^{\rm i})\big)+\xi^j_\mu\big(X_2^\mu(\sigma_j^{\rm i})-X^\mu_{\rm sBCL}(\sigma_j^{\rm i})\big)$$
$$+\tilde\kappa^j_\mu\big(X_1^\mu(\sigma_j^{\rm f})-X^\mu_{\rm sBCR}(\sigma_j^{\rm f})\big)+\tilde\xi^j_\mu\big(X_2^\mu(\sigma_j^{\rm f})-X^\mu_{\rm sBCR}(\sigma_j^{\rm f})\big)$$
$$+\kappa^j_\phi\big(\phi_1(\sigma_j^{\rm i})-\phi_{\rm sBCR}(\sigma_j^{\rm i})\big)+\xi^j_\phi\big(\phi_2(\sigma_j^{\rm i})-\phi_{\rm sBCR}(\sigma_j^{\rm i})\big)$$
$$+\tilde\kappa^j_\phi\big(\phi_1(\sigma_j^{\rm f})-\phi_{\rm sBCL}(\sigma_j^{\rm f})\big)+\tilde\xi^j_\phi\big(\phi_2(\sigma_j^{\rm f})-\phi_{\rm sBCL}(\sigma_j^{\rm f})\big)\Big\},$$

$$=\int d^{(d+1)}\Sigma\,\Big\{E^{\rm L}_{\rm BVP}[X_1,\partial_a X_1,\phi_1,\partial_a\phi_1]-E^{\rm L}_{\rm BVP}[X_2,\partial_a X_2,\phi_2,\partial_a\phi_2]\Big\}$$

Forward & backward branch Lagrangian

Initial conditions for coordinate maps and fields

Connecting conditions for maps and fields from calssical Schwinger-Keldysh

Spatial boundary conditions for the coordinate maps and fields

Redefined Lagrangians including Lagrange mult.