Tom & Jerry triples unprojection format

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Fano varieties in Stavanger,
April 02, 2025

MAIN GOALS OF THE PRESENT TALK:

1) Introduction of Tom and Jerry triples unprojection format.

2) Use Tom and Jerry triples unprojection format for the construction of two families of codimension 6 Fano 3-folds described in the Graded Ring Database.

PRELIMINARIES

ASSUMPTION All rings are commutative and with unit.

DEFINITION Assume $A = [a_{ij}]$ is an $m \times m$ skewsymmetric matrix,
(i.e., $a_{ji} = -a_{ij}$ and $a_{ii} = 0$) with entries in a ring R.

- If $m = 2\ell$ then $\det A = f(a_{ij})^2$.
  The polynomial $f(a_{ij})$ is called the Pfaffian of the matrix A and is denoted by $Pf(A)$.
- If $m = 2\ell + 1$ by Pfaffians of A we mean the set

$$\{Pf(A_1), Pf(A_2), \ldots, Pf(A_m)\},$$

  where A_i denotes the skewsymmetric submatrix of A obtained by deleting the ith row and ith column of A.

EXAMPLE

- For $m = 2$:

$$Pf(\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix}) = a_{12}$$

- For $m = 5$:

$$Pf(\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{pmatrix}) =$$

$$= \quad \{Pf(A_1), Pf(A_2), \ldots, Pf(A_5)\}$$

where,

$$Pf(A_1) = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34},$$

$$Pf(A_2) = a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34},$$

$$Pf(A_3) = a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24},$$

$$Pf(A_4) = a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23},$$

$$Pf(A_5) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

DEFINITION A Noetherian local ring R is a Gorenstein ring if $\operatorname{inj}\dim_R R < \infty$.
More generally, a Noetherian ring R is called Gorenstein if for every maximal ideal $\mathfrak{m}$ of R the localization $R_{\mathfrak{m}}$ is Gorenstein.

EXAMPLES OF GORENSTEIN RINGS

- The anticanonical ring $R = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(-mK_X))$ of a (smooth) Fano n-fold.
- The canonical ring $R = \bigoplus_{m\geq 0} H^0(X, \mathcal{O}_X(mK_X))$ of a (smooth) regular surface of general type.
- The Stanley-Reisner ring of a simplicial sphere over any field.

THEOREM Let $R = k[x_1, \ldots, x_m]/I$ be the polynomial ring in n variables divided by a homogeneous ideal I.

- (Serre) If codim I $=$ 1 or 2 then

  $$R \textit{ is Gorenstein} \Leftrightarrow I \textit{ is a complete intersection.}$$

- (Buchsbaum-Eisenbud (1977)) If codim I $= 3$ then R is Gorenstein $\Leftrightarrow$ I is generated by the $2n \times 2n$ Pfaffians of a skewsymmetric $(2n+1) \times (2n+1)$ matrix with entries in $k[x_1, \ldots, x_m]$.

QUESTION Is there a structure theorem for codim I ≥ 4 ?

- A.Kustin & M.Milller (1983) introduced a procedure which constructs more «complicated» Gorenstein rings from simpler ones by increasing codimension. This procedure is called Kustin-Miller unprojection.
- M.Reid (1995) rediscovered what was essentially the same procedure working with Gorenstein rings arising from K3 surfaces and 3-folds.

UNPROJECTION REVIEW

Kustin-Miller unprojection

Assumptions of Kustin-Miller unprojection:

- $J \subset R$ codimension 1 ideal
- R Gorenstein
- R/J Gorenstein.

Codimension: increasing by one.

Applying the functor $\mathrm{Hom}_R(-, R)$ to the following short exact sequence

$$0 \to J \to R \to R/J \to 0$$

we get a corresponding long exact sequence.

Using duality theory, we obtain the exact sequence

$$0 \to R \to \mathrm{Hom}_R(J, R) \to R/J \to 0$$

with the last nonzero map corresponding to the Poincaré residue map of complex geometry.

Hence, there exists $\phi \in \mathrm{Hom}_R(J, R)$ such that together with the inclusion $i : J \to R$ generate the R-module $\mathrm{Hom}_R(J, R)$.

DEFINITION (Reid) The Kustin-Miller unprojection ring of the pair $J \subset R$ is the ring

$$\mathrm{Unpr}(J, R) = \textit{graph of } \phi = \frac{R[T]}{(T\alpha - \phi(\alpha) : \ \alpha \in J)}$$

where T is a new variable.

THEOREM (Kustin-Miller, Reid-Papadakis) The ring $\mathrm{Unpr}(J, R)$ is Gorenstein.

REMARK: We have that $\mathrm{Unpr}(J, R)$ has typically more complicated structure than both $R, R/J$ and is useful to construct/analyse Gorenstein rings in terms of simpler ones.

Parallel Kustin-Miller unprojection

Kustin-Miller unprojection can be used many times in an inductive way to produce Gorenstein rings of arbitrary codimension, whose properties are nevertheless controlled by just a few equations as a number of new unprojection variables are adjoined.

APPLICATIONS

- Construction of new interesting algebraic surfaces and 3-folds.
- Explicit Birational Geometry.
  (That is, writing down explicitly varieties, morphisms and rational maps that Minimal Model Program says they exist.)
- Algebraic Combinatorics.

Neves and Papadakis (2013) develop a theory, which is called parallel Kustin-Miller unprojection.

They set sufficient conditions on a positively graded Gorenstein ring R and a finite set of codimension 1 ideals which ensure the series of unprojections.

Furthermore, they give a simple and explicit description of the end product ring which corresponds to the unprojection of the ideals.

This theory applies when all the unprojection ideals of a series of unprojections correspond to ideals already present in the initial ring.

TOM & JERRY TRIPLES

Assume J is a codimension 4 complete intersection ideal and M is a 5×5 skewsymmetric matrix.

DEFINITION

1. Assume $1 \leq i \leq 5$. The matrix M is called Tom_i in J if after we delete the i-th row and i-th column of M the remaining entries are elements of the codimension 4 ideal J.
2. Assume $1 \leq i < j \leq 5$. The matrix M is called $Jerry_{ij}$ in J if all the entries of M that belong to the i-th row or i-th column or j-th row or j-th column are elements of J.

REMARK In both cases the Pfaffian ideal of M is a subset of J.

Papadakis' Calculation for Tom (2004)

Let $R = k[x_k, z_k, m_{ij}^k]$, where $1 \leq k \leq 4$, $2 \leq i < j \leq 5$, be a polynomial ring. Set $J = (z_1, z_2, z_3, z_4)$. Denote by

$$N = \begin{pmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ -x_1 & 0 & m_{23} & m_{24} & m_{25} \\ -x_2 & -m_{23} & 0 & m_{34} & m_{35} \\ -x_3 & -m_{24} & -m_{34} & 0 & m_{45} \\ -x_4 & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix},$$

where

$$m_{ij} = \sum_{k=1}^{4} m_{ij}^k z_k.$$

Let I be the ideal generated by the Pfaffians P_0, P_1, P_2, P_3, P_4 of N. It holds that $I \subset J$.

Papadakis using multilinear and homological algebra calculated the equations of the codimension 4 ring which occurs as unprojection of the pair $I \subset J$.

More precisely, he calculated 4 polynomials g_i for i $= 1, \dots, 4$ and defines the map ϕ by

$$\phi \colon J/I \to R/I, \quad z_i + I \mapsto g_i + I.$$

Moreover, he proved that $\text{Hom}_{R/I}(J/I, R/I)$ is generated as R/I-module by the inclusion map i and ϕ. From the theory it follows that the ideal

$$(P_0, P_1, P_2, P_3, P_4, Tz_1 - g_1, Tz_2 - g_2, Tz_3 - g_3, Tz_4 - g_4)$$

of the polynomial ring $R[T]$ is Gorenstein of codimension 4.

We will now define Tom & Jerry triples.

Let

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & m_{34} & m_{35} \\ -m_{14} & -m_{24} & -m_{34} & 0 & m_{45} \\ -m_{15} & -m_{25} & -m_{35} & -m_{45} & 0 \end{pmatrix}$$

be a 5×5 skewsymmetric matrix and J_1, J_2, J_3 be three complete intersection ideals of codimension 4.

Tom & Tom & Tom case

DEFINITION We say that M is a $\text{Tom}_1 + \text{Tom}_2 + \text{Tom}_3$ in J_1, J_2, J_3 if the entries of M satisfy the following conditions:

$$m_{12} \in J_3,\ m_{13} \in J_2,\ m_{14}, m_{15} \in J_2 \cap J_3,\ m_{23} \in J_1,$$
$$m_{24}, m_{25} \in J_1 \cap J_3,\ m_{34}, m_{35} \in J_1 \cap J_2,\ m_{45} \in J_1 \cap J_2 \cap J_3.$$

REMARK Equivalently, the matrix M is Tom_1 in J_1, Tom_2 in J_2 and Tom_3 in J_3.

Similarly, we set conditions in the entries of M such that M is

- Jerry_{ij} in J_1, Jerry_{kl} in J_2 and Jerry_{mn} in J_3.
- Tom_i in J_1, Tom_j in J_2 and Jerry_{kl} in J_3.
- Tom_i in J_1, Jerry_{jk} in J_2 and Jerry_{lm} in J_3.

We work over the polynomial ring $R = k[z_i, c_j]$, where $1 \le i \le 7$ and $1 \le j \le 25$. Denote by Tom(1,2,3), the following 5×5 skewsymmetric matrix

$$\begin{pmatrix} 0 & c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_6 & c_5 z_1 + c_6 z_2 + c_7 z_4 + c_8 z_5 & c_9 z_1 + c_{10} z_2 & c_{11} z_1 + c_{12} z_2 \\ & 0 & c_{13} z_2 + c_{14} z_3 + c_{15} z_5 + c_{16} z_7 & c_{17} z_2 + c_{18} z_3 & c_{19} z_2 + c_{20} z_3 \\ & & 0 & c_{21} z_2 + c_{22} z_5 & c_{23} z_2 + c_{24} z_5 \\ & -Sym & & 0 & c_{25} z_2 \\ & & & & 0 \end{pmatrix}$$

which is $\mathrm{Tom}_1 + \mathrm{Tom}_2 + \mathrm{Tom}_3$ matrix in the ideals

$$J_1 = (z_2, z_3, z_5, z_7), \quad J_2 = (z_1, z_2, z_4, z_5), \quad J_3 = (z_1, z_2, z_3, z_6).$$

Let I be the ideal generated by the Pfaffians of Tom(1,2,3).

PROPOSITION

(i) For all t with $1 \leq t \leq 3$, the ideal J_t/I is a codimension 1 homogeneous ideal of R/I with Gorenstein quotient.

(ii) For all t, s with $1 \leq t < s \leq 3$, it holds that

$$codim_{R/I}(J_t/I + J_s/I) = 3.$$

AIM: Compution of $\phi_t \colon J_t/I \to R/I$ for all t with $1 \leq t \leq 3$.

STRATEGY: We combine Papadakis' Calculation for Tom_1 with the fact that a Tom_i matrix in an ideal J is related to Tom_1 matrix in the ideal J via a sequence of elementary row and column operations.

PROPOSITION For all t with $1 \leq t \leq 3$, the R/I-module $\mathrm{Hom}_{R/I}(J_t/I, R/I)$ is generated by the two elements i_t and ϕ_t.

PROPOSITION For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, it holds that

$$\phi_s(J_s/I) \subset J_t/I.$$

PROPOSITION For all t, s with $1 \leq t, s \leq 3$ and $t \neq s$, there exists a homogeneous element A_{st} such that

$$\phi_s(\phi_t(p)) = A_{st} p \textit{ for all } p \in J_t/I.$$

Let T_1, T_2, T_3 be three new variables of degree 6.

DEFINITION We define as I_{un} the ideal

$$(I) + (T_1 z_2 - \phi_1(z_2),\ T_1 z_3 - \phi_1(z_3),\ T_1 z_5 - \phi_1(z_5),\ T_1 z_7 - \phi_1(z_7),$$

$$T_2 z_1 - \phi_2(z_1),\ T_2 z_2 - \phi_2(z_2),\ T_2 z_4 - \phi_2(z_4), T_2 z_5 - \phi_2(z_5),$$

$$T_3 z_1 - \phi_3(z_1),\ T_3 z_2 - \phi_3(z_2),\ T_3 z_3 - \phi_3(z_3),\ T_3 z_6 - \phi_3(z_6),$$

$$T_1 T_2 - A_{12},\ T_1 T_3 - A_{13},\ T_2 T_3 - A_{23})$$

of the polynomial ring $R[T_1, T_2, T_3]$.

We set $R_{un} = R[T_1, T_2, T_3]/I_{un}$.

PROPOSITION The homogeneous ideal I_{un} is a codimension 6 ideal with a minimal generating set of 20 elements.

THEOREM (P.) The ring R_{un} is Gorenstein.

APPLICATIONS

We now give two applications of the construction of R_{un}.

Theorem (P.) There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano 3-folds $X \subset \mathbb{P}(1^3, 2^7)$, nonsingular away from eight quotient singularities $\frac{1}{2}(1,1,1)$, with Hilbert series of the anticanonical ring

$$\frac{1 - 20t^4 + 64t^6 - 90t^8 + 64t^{10} - 20t^{12} + t^{16}}{(1-t)^3(1-t^2)^7}.$$

Theorem (P.) There exists a family of quasismooth, projectively normal and projectively Gorenstein Fano 3-folds $X \subset \mathbb{P}(1^3, 2^5, 3^2)$, nonsingular away from four quotient singularities $\frac{1}{2}(1,1,1)$, and two quotient singularities $\frac{1}{3}(1,1,2)$, with Hilbert series of the anticanonical ring

$$\frac{1 - 11t^4 - 8t^5 + 23t^6 + 32t^7 - 13t^8 - 48t^9 - 13t^{10} + 32t^{11} + 23t^{12} - 8t^{13} - 11t^{14} + t^{18}}{(1-t)^3(1-t^2)^5(1-t^3)^2}.$$

Construction of the first family:

Denote by $k = \mathbb{C}$ the field of complex numbers.
Let R_{un} be the ring and I_{un} the ideal which were defined above.
Substitute the variables $(c_1, \dots, c_{25})$ with a general element of k^{25}.
$\hat{R}_{un}$: the ring which occurs from R_{un} after this substitution.
$\hat{I}_{un}$: the ideal which obtained by the ideal I_{un} after this substitution.
In what follows we set

$$degree\ z_i = degree\ T_1 = degree\ T_2 = degree\ T_3 = 2,$$

for all i with $1 \leq i \leq 7$.
Since R_{un} is Gorenstein, Proj $\hat{R}_{un} \subset \mathbb{P}(2^{10})$ is a projectively Gorenstein 3-fold.

Let $A = k[w_1, w_2, w_3, z_1, z_2, z_3, z_5, T_1, T_2, T_3]$ be the polynomial ring over k with w_1, w_2, w_3 variables of degree 1. Consider the graded k-algebra homomorphism

$$\psi\colon \hat{R}_{un}[T_1, T_2, T_3] \to A$$

with

$$\psi(z_1) = z_1, \quad \psi(z_2) = z_2, \quad \psi(z_3) = z_3, \quad \psi(z_4) = f_1,$$

$$\psi(z_5) = z_5, \quad \psi(z_6) = f_2, \quad \psi(z_7) = f_3, \quad \psi(T_1) = T_1,$$

$$\psi(T_2) = T_2, \quad \psi(T_3) = T_3$$

where
$f_1 = l_1 z_1 + l_2 z_2 + l_3 z_3 + l_4 z_5 + l_5 T_1 + l_6 T_2 + l_7 T_3 + l_8 w_1^2 + l_9 w_1 w_2 + l_{10} w_1 w_3 + l_{11} w_2^2 + l_{12} w_2 w_3 + l_{13} w_3^2,$

$$f_2 = l_{14} z_1 + l_{15} z_2 + l_{16} z_3 + l_{17} z_5 + l_{18} T_1 + l_{19} T_2 + l_{20} T_3 + l_{21} w_1^2 + l_{22} w_1 w_2 + l_{23} w_1 w_3 + l_{24} w_2^2 + l_{25} w_2 w_3 + l_{26} w_3^2,$$

$$f_3 = l_{27} z_1 + l_{28} z_2 + l_{29} z_3 + l_{30} z_5 + l_{31} T_1 + l_{32} T_2 + l_{33} T_3 + l_{34} w_1^2 + l_{35} w_1 w_2 + l_{36} w_1 w_3 + l_{37} w_2^2 + l_{38} w_2 w_3 + l_{39} w_3^2$$

and $(l_1, \ldots, l_{39}) \in k^{39}$ are general.

Denote by Q the ideal of the ring A generated by the subset $\psi(\hat{I}_{un})$. Let $X = V(Q) \subset \mathbb{P}(1^3, 2^7)$. Then X is a codimension 6 projectively Gorenstein 3-fold.

PROPOSITION The ring A/Q is an integral domain.

PROPOSITION Consider $X = V(Q) \subset \mathbb{P}(1^3, 2^7)$. Denote by $X_{cone} \subset \mathbb{A}^{10}$ the affine cone over X. The scheme X_{cone} is smooth outside the vertex of the cone.

PROPOSITION Consider the singular locus $Z = V(w_1, w_2, w_3)$ of the weighted projective space $\mathbb{P}(1^3, 2^7)$. The intersection of X with Z consists of exactly eight points which are quotient singularities of type $\frac{1}{2}(1,1,1)$ for X.

PROPOSITION The minimal graded resolution of A/Q as A-module is equal to

$$0 \to A(-16) \to A(-12)^{20} \to A(-10)^{64} \to A(-8)^{90} \to A(-6)^{64}$$
$$\to A(-4)^{20} \to A$$

Moreover, the canonical module of A/Q is isomorphic to $(A/Q)(-1)$ and the Hilbert series of A/Q as graded A-module is equal to

$$\frac{1-20t^4+64t^6-90t^8+64t^{10}-20t^{12}+t^{16}}{(1-t)^3(1-t^2)^7}.$$

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Thank you for your attention!!