The Research Council of Norway

University of Stavanger

Recent progress on quarkonium real-time dynamics

Alexander Rothkopf
Faculty of Science and Technology
Department of Mathematics and Physics
University of Stavanger

References:

A.R. ”Heavy Quarkonium in Extreme Conditions” Phys.Rept. 858 (2020)
O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. arXiv:2004.04406
A. Lehmann, A.R. arXiv:2003.02509 (& in preparation)

Overview

University of Stavanger

- **Motivation**
- **A new technique to accurately solve Quarkonium dynamics**
  O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. arXiv:2004.04406
- **A solution to a puzzle on heavy quark interactions**
  A. Lehmann, A.R. arXiv:2003.02509 (& in preparation)
- **Conclusion**

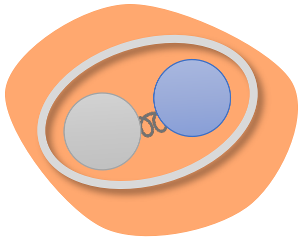
Quarkonium - Motivation

a clean QCD laboratory

Theory advantage: separation of scales enables powerful effective field theory tools

a precision probe in HIC studies

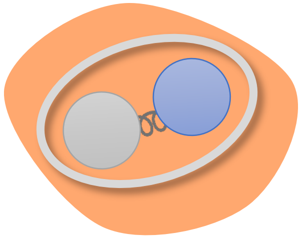
Experiment advantage: clean signals via enhanced dilepton decay channels



Quarkonium - Motivation

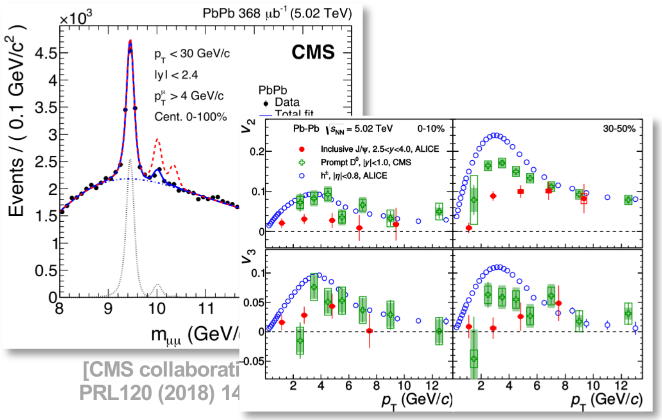
a clean QCD laboratory

Theory advantage: separation of scales enables powerful effective field theory tools



a precision probe in HIC studies

Experiment advantage: clean signals via enhanced dilepton decay channels



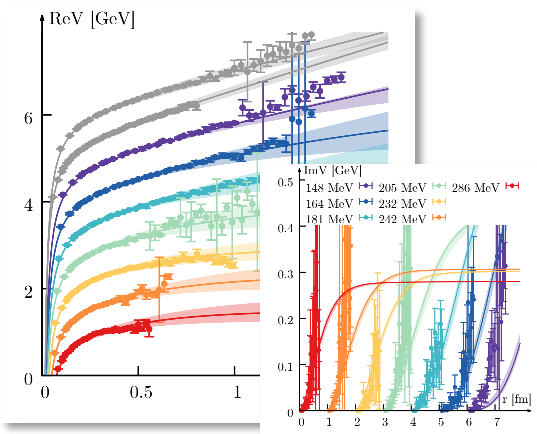
[CMS collaborati
PRL120 (2018) 14

[ALICE collaboration]
JHEP 1902 (2019) 012

Quarkonium - Motivation

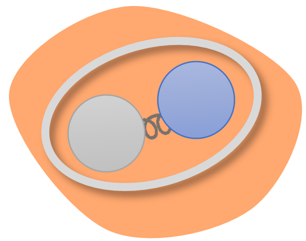
a clean QCD laboratory

Theory advantage: separation of scales enables powerful effective field theory tools



Y.Burnier, O.Kaczmarek, A.R. PRL114 (2015) 082001
D. Lafferty, A.R., arXiv:1906.00035v3

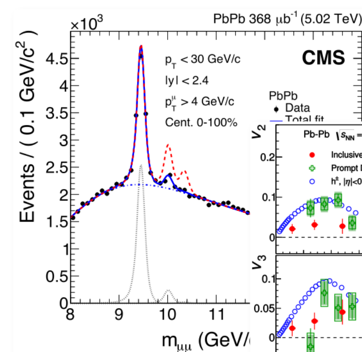
1st principles + intuitive non-relativistic language (e.g. via a non-relativistic potential)



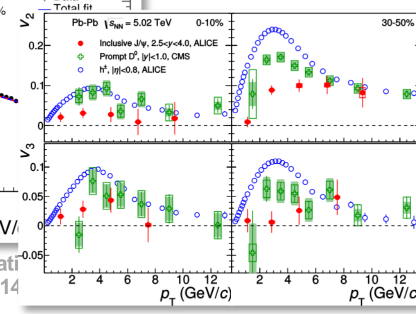
$$\frac{\Lambda_{QCD}}{m_Q} \ll 1 \qquad \frac{\epsilon_{env}}{m_Q} \ll 1$$

a precision probe in HIC studies

Experiment advantage: clean signals via enhanced dilepton decay channels



[CMS collaborati
PRL120 (2018) 14



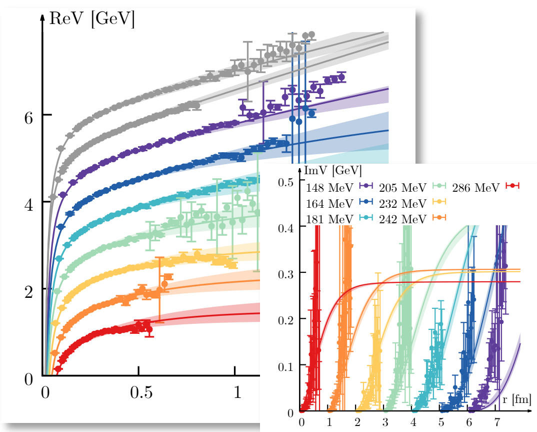
[ALICE collaboration]
JHEP 1902 (2019) 012

Quarkonium - Motivation

University of Stavanger

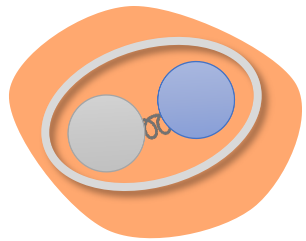
a clean QCD laboratory

Theory advantage: separation of scales enables powerful effective field theory tools



Y.Burnier, O.Kaczmarek, A.R. PRL114 (2015) 082001
D. Laffefrty, A.R., arXiv:1906.00035v3

1st principles + intuitive non-relativistic language (e.g. via a non-relativistic potential)

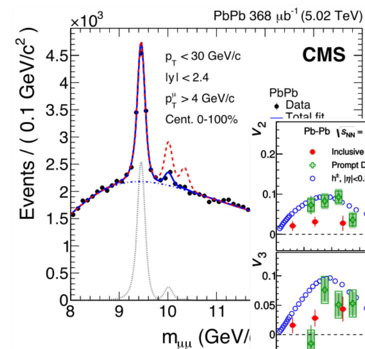


$$\frac{\Lambda_{\mathrm{QCD}}}{m_Q} \ll 1 \qquad \frac{\varepsilon_{\mathrm{env}}}{m_Q} \ll 1$$

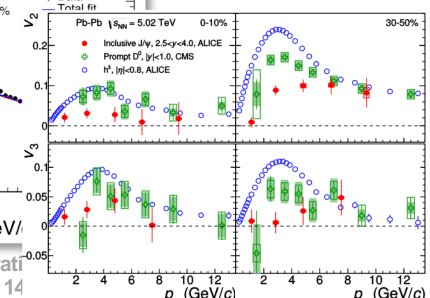
For a comprehensive review see: A.R. "Heavy Quarkonium in Extreme Conditions" Phys.Rept. 858 (2020)

a precision probe in HIC studies

Experiment advantage: clean signals via enhanced dilepton decay channels



[CMS collaborati
PRL120 (2018) 14



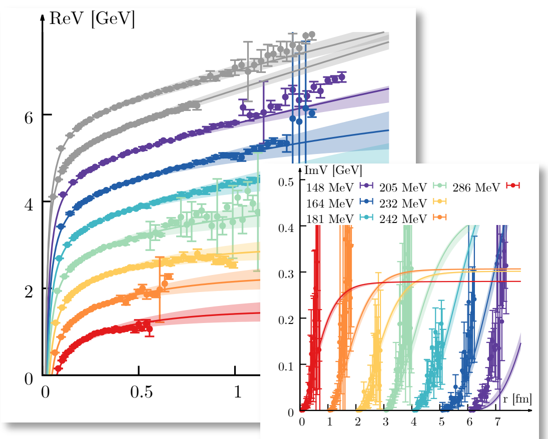
[ALICE collaboration]
JHEP 1902 (2019) 012

Quarkonium - Motivation

University of Stavanger

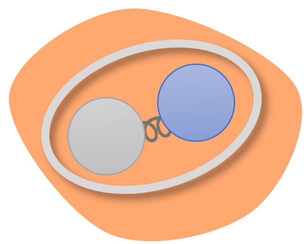
a clean QCD laboratory

Theory advantage: separation of scales enables powerful effective field theory tools



Y.Burnier, O.Kaczmarek, A.R. PRL114 (2015) 082001
D. Laffefrty, A.R., arXiv:1906.00035v3

1st principles + intuitive non-relativistic language (e.g. via a non-relativistic potential)

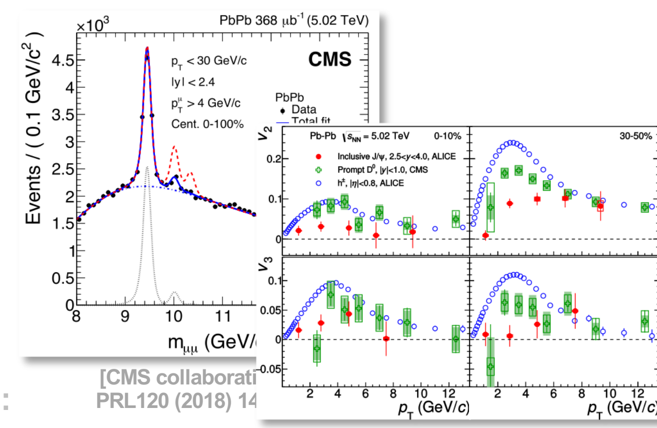


$$\frac{\Lambda_{\rm QCD}}{m_Q} \ll 1 \qquad \frac{\varepsilon_{\rm env}}{m_Q} \ll 1$$

For a comprehensive review see:
A.R. "Heavy Quarkonium in Extreme Conditions"
Phys.Rept. 858 (2020)

a precision probe in HIC studies

Experiment advantage: clean signals via enhanced dilepton decay channels



[CMS collaborati
PRL120 (2018) 14

[ALICE collaboration]
JHEP 1902 (2019) 012

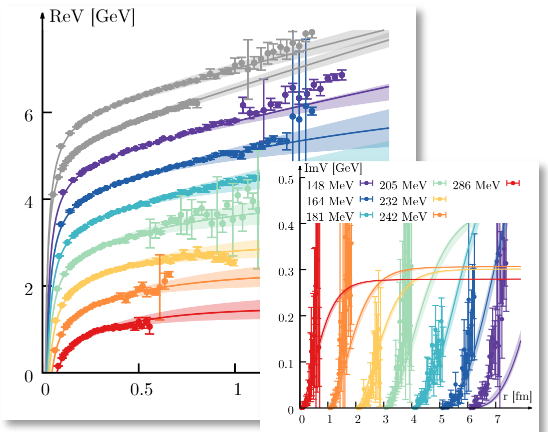
- Challenge I: Derive a real-time description from QCD and solve it numerically

Quarkonium - Motivation

University of Stavanger

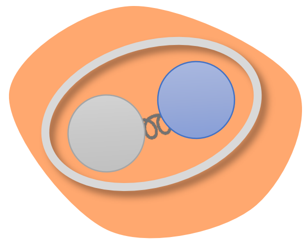
a clean QCD laboratory

Theory advantage: separation of scales enables powerful effective field theory tools



Y.Burnier, O.Kaczmarek, A.R. PRL114 (2015) 082001
D. Laffefrty, A.R., arXiv:1906.00035v3

1st principles + intuitive non-relativistic language (e.g. via a non-relativistic potential)

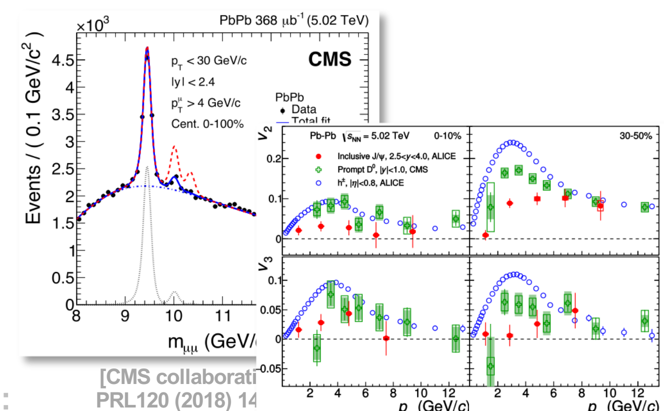


$$\frac{\Lambda_{QCD}}{m_Q} \ll 1 \qquad \frac{\varepsilon_{env}}{m_Q} \ll 1$$

For a comprehensive review see:
A.R. "Heavy Quarkonium in Extreme Conditions"
Phys.Rept. 858 (2020)

a precision probe in HIC studies

Experiment advantage: clean signals via enhanced dilepton decay channels



[CMS collaboration] PRL120 (2018) 14...

[ALICE collaboration] JHEP 1902 (2019) 012

- Challenge I: Derive a real-time description from QCD and solve it numerically
- Challenge II: Establish under which conditions potential picture is applicable

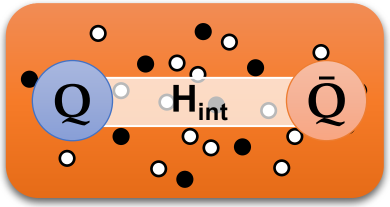
Challenge I: Derive a real-time description from QCD and solve it numerically

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H,\rho]$$

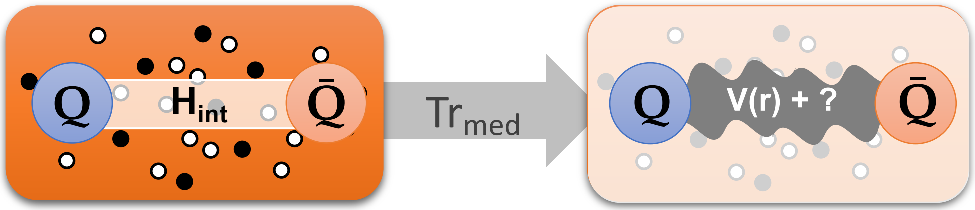


$$H_{\rm int} = \sum_m \Sigma_m \otimes \Xi_m$$

The open quantum systems picture

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H,\rho]$$



$$H_{\rm int} = \sum_m \Sigma_m \otimes \Xi_m$$

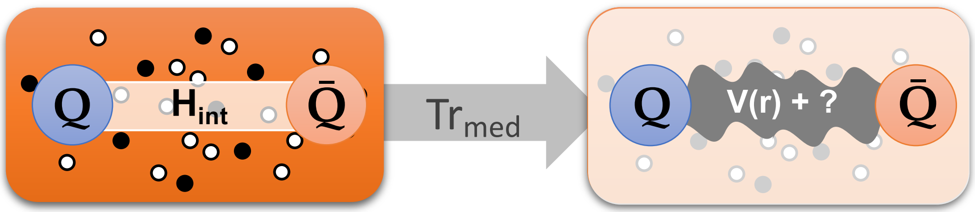
  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = {\rm Tr}_{med}\left[\rho\right]$ $\quad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H, \rho]$$



$$H_{\rm int} = \sum_m \Sigma_m \otimes \Xi_m$$

  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = \mathrm{Tr}_{med}\big[\rho\big]$ $\qquad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

- **Separation of time-scales** determines the nature of the e.o.m. :

Environment relaxation scale τ_E : $\langle \Xi_m(t)\Xi_m(0)\rangle \sim e^{-t/\tau_E}$

$Q\bar{Q}$ system scale τ_S : $\tau_S \sim 1/|\omega - \omega'|$

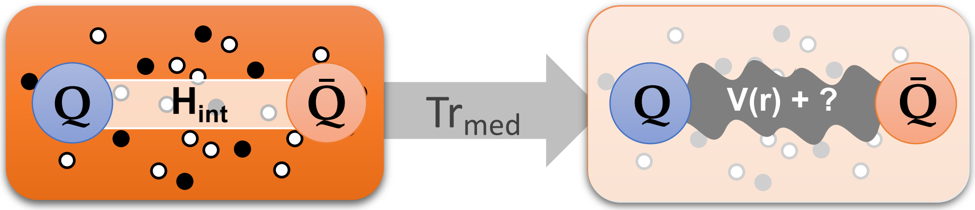
$Q\bar{Q}$ relaxation scale $\tau_{\rm rel}$: $\langle p(t)\rangle \propto e^{-t/\tau_{\rm rel}}$

The open quantum systems picture

University of Stavanger

- Need a general approach to describe quarkonium coupled to a thermal medium
  - Overall system is closed, hermitean Hamiltonian: von Neumann equation

$$H = H_{Q\bar{Q}} \otimes I_{med} + I_{Q\bar{Q}} \otimes H_{med} + H_{int} \qquad \frac{d\rho}{dt} = -i[H,\rho]$$



$$H_{\rm int} = \sum_m \Sigma_m \otimes \Xi_m$$

  - Dynamics of the reduced QQ system: $\rho_{Q\bar{Q}} = {\rm Tr}_{med}\big[\rho\big] \qquad \frac{d}{dt}\rho_{Q\bar{Q}} = \mathcal{V}\rho_{Q\bar{Q}}$

- **Separation of time-scales** determines the nature of the e.o.m. :

Environment relaxation scale τ_E : $\langle \Xi_m(t)\Xi_m(0)\rangle \sim e^{-t/\tau_E}$

$Q\bar{Q}$ system scale τ_S : $\tau_S \sim 1/|\omega - \omega'|$

$Q\bar{Q}$ relaxation scale $\tau_{\rm rel}$: $\langle p(t)\rangle \propto e^{-t/\tau_{\rm rel}}$

- In case of Markovian time evolution ($\tau_E \ll \tau_{\rm rel}$) leads to a **Lindblad equation**

$$\frac{d}{dt}\rho_{Q\bar{Q}} = -i[\tilde{H}_{Q\bar{Q}}, \rho_{Q\bar{Q}}] + \sum_k \gamma_k \Big(L_k \rho_{Q\bar{Q}} L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho_{Q\bar{Q}} - \frac{1}{2}\rho_{Q\bar{Q}} L_k^\dagger L_k \Big)$$

Solving the Lindblad equation

University of Stavanger

- Preservation of the continuum properties?

$$\langle\psi_n^{\rm S}|\rho_{\rm S}|\psi_n^{\rm S}\rangle > 0, \forall n, \quad \rho_{\rm S}^\dagger = \rho_{\rm S}, \quad {\rm Tr}[\rho_{\rm S}] = 1$$

1d Lindblad equation in coord. space

$$\begin{aligned}
\partial_t\rho^{\rm rel}(x,y,t) = i\Big[\frac{1}{M}\frac{\partial^2}{\partial x^2} - V(x)\Big]\rho^{\rm rel}(x,y,t) - i\Big[\frac{1}{M}\frac{\partial^2}{\partial y^2} - V(y)\Big]\rho^{\rm rel}(x,y,t) \qquad (20)\\
+\Big[2F_1\Big(\frac{x-y}{2}\Big) - 2F_1(0) + F_1(x) + F_1(y) - 2F_1\Big(\frac{x+y}{2}\Big)\Big]\rho^{\rm rel}(x,y,t)\\
-\Big[\frac{\partial^2}{\partial x^2}\frac{\frac{\partial^2}{\partial x^2}A(x)}{4M^2} + \frac{\partial^2}{\partial y^2}\frac{\frac{\partial^2}{\partial y^2}A(y)}{4M^2}\Big]\rho^{\rm rel}(x,y,t)\\
+\Big[2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial x}\frac{\frac{\partial^2}{\partial x^2}A(x)}{M^2}\Big]\frac{\partial}{\partial x}\rho^{\rm rel}(x,y,t)\\
+\Big[-2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial y}\frac{\frac{\partial^2}{\partial y^2}A(y)}{M^2}\Big]\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
+\Big[2F_3\Big(\frac{x-y}{2}\Big) + 2F_3\Big(\frac{x+y}{2}\Big)\Big]\frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
+\Big[F_3(0) + F_3(x)\Big]\frac{\partial^2}{\partial x^2}\rho^{\rm rel}(x,y,t) + \Big[F_3(0) + F_3(y)\Big]\frac{\partial^2}{\partial y^2}\rho^{\rm rel}(x,y,t).
\end{aligned}$$

$$F_1(x) = \Big[D(x) + \frac{1}{4MT}\frac{\partial^2}{\partial x^2}D(x) + \frac{1}{8M^2}\frac{\partial^4}{\partial x^4}A(x)\Big],$$
$$F_2(x) = \frac{1}{4MT}\frac{\partial}{\partial x}D(x) + \frac{1}{4M^2}\frac{\partial^3}{\partial x^3}A(x), \quad F_3(x) = -\frac{1}{2M^2}\frac{\partial^2}{\partial x^2}A(x)$$

Solving the Lindblad equation

University of Stavanger

- Preservation of the continuum properties?

$$\langle\psi_n^{\rm S}|\rho_{\rm S}|\psi_n^{\rm S}\rangle > 0, \forall n, \quad \rho_{\rm S}^\dagger = \rho_{\rm S}, \quad {\rm Tr}[\rho_{\rm S}] = 1$$

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C},$$
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

1d Lindblad equation in coord. space

$$\begin{aligned}
\partial_t \rho^{\rm rel}(x,y,t) &= i\Big[\frac{1}{M}\frac{\partial^2}{\partial x^2} - V(x)\Big]\rho^{\rm rel}(x,y,t) - i\Big[\frac{1}{M}\frac{\partial^2}{\partial y^2} - V(y)\Big]\rho^{\rm rel}(x,y,t) \qquad (20)\\
&+\Big[2F_1\Big(\frac{x-y}{2}\Big) - 2F_1(0) + F_1(x) + F_1(y) - 2F_1\Big(\frac{x+y}{2}\Big)\Big]\rho^{\rm rel}(x,y,t)\\
&-\Big[\frac{\partial^2}{\partial x^2}\frac{\frac{\partial^2}{\partial x^2}A(x)}{4M^2} + \frac{\partial^2}{\partial y^2}\frac{\frac{\partial^2}{\partial y^2}A(y)}{4M^2}\Big]\rho^{\rm rel}(x,y,t)\\
&+\Big[2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial x}\frac{\frac{\partial^2}{\partial x^2}A(x)}{M^2}\Big]\frac{\partial}{\partial x}\rho^{\rm rel}(x,y,t)\\
&+\Big[-2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial y}\frac{\frac{\partial^2}{\partial y^2}A(y)}{M^2}\Big]\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
&+\Big[2F_3\Big(\frac{x-y}{2}\Big) + 2F_3\Big(\frac{x+y}{2}\Big)\Big]\frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
&+\Big[F_3(0) + F_3(x)\Big]\frac{\partial^2}{\partial x^2}\rho^{\rm rel}(x,y,t) + \Big[F_3(0) + F_3(y)\Big]\frac{\partial^2}{\partial y^2}\rho^{\rm rel}(x,y,t).
\end{aligned}$$

$$\begin{aligned}
F_1(x) &= \Big[D(x) + \frac{1}{4MT}\frac{\partial^2}{\partial x^2}D(x) + \frac{1}{8M^2}\frac{\partial^4}{\partial x^4}A(x)\Big],\\
F_2(x) &= \frac{1}{4MT}\frac{\partial}{\partial x}D(x) + \frac{1}{4M^2}\frac{\partial^3}{\partial x^3}A(x), \quad F_3(x) = -\frac{1}{2M^2}\frac{\partial^2}{\partial x^2}A(x)
\end{aligned}$$

Solving the Lindblad equation

University of Stavanger

- Preservation of the continuum properties?

$$\langle \psi_n^{\mathrm{S}} | \rho_{\mathrm{S}} | \psi_n^{\mathrm{S}} \rangle > 0, \forall n, \quad \rho_{\mathrm{S}}^\dagger = \rho_{\mathrm{S}}, \quad \mathrm{Tr}[\rho_{\mathrm{S}}] = 1$$

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C},$$

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

- Starting point: Crank-Nicholson scheme
  (which for non-dissipative dynamics preserves exactly)

$$\rho(x, y, t + \Delta t) = exp[-i\Delta t \mathcal{L}]\rho(x, y, t) \approx \frac{1 - \frac{1}{2} i \Delta t \mathcal{L}}{1 + \frac{1}{2} i \Delta t \mathcal{L}} \rho(x, y, t)$$

1d Lindblad equation in coord. space

$$\begin{aligned}
\partial_t \rho^{\mathrm{rel}}(x,y,t) = {} & i\Big[\frac{1}{M}\frac{\partial^2}{\partial x^2} - V(x)\Big]\rho^{\mathrm{rel}}(x,y,t) - i\Big[\frac{1}{M}\frac{\partial^2}{\partial y^2} - V(y)\Big]\rho^{\mathrm{rel}}(x,y,t) \qquad (20)\\
& + \Big[2F_1\Big(\frac{x-y}{2}\Big) - 2F_1(0) + F_1(x) + F_1(y) - 2F_1\Big(\frac{x+y}{2}\Big)\Big]\rho^{\mathrm{rel}}(x,y,t)\\
& - \Big[\frac{\partial^2}{\partial x^2}\frac{\frac{\partial^2}{\partial x^2}A(x)}{4M^2} + \frac{\partial^2}{\partial y^2}\frac{\frac{\partial^2}{\partial y^2}A(y)}{4M^2}\Big]\rho^{\mathrm{rel}}(x,y,t)\\
& + \Big[2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial x}\frac{\frac{\partial^2}{\partial x^2}A(x)}{M^2}\Big]\frac{\partial}{\partial x}\rho^{\mathrm{rel}}(x,y,t)\\
& + \Big[-2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial y}\frac{\frac{\partial^2}{\partial y^2}A(y)}{M^2}\Big]\frac{\partial}{\partial y}\rho^{\mathrm{rel}}(x,y,t)\\
& + \Big[2F_3\Big(\frac{x-y}{2}\Big) + 2F_3\Big(\frac{x+y}{2}\Big)\Big]\frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho^{\mathrm{rel}}(x,y,t)\\
& + \Big[F_3(0) + F_3(x)\Big]\frac{\partial^2}{\partial x^2}\rho^{\mathrm{rel}}(x,y,t) + \Big[F_3(0) + F_3(y)\Big]\frac{\partial^2}{\partial y^2}\rho^{\mathrm{rel}}(x,y,t).
\end{aligned}$$

$$F_1(x) = \Big[D(x) + \frac{1}{4MT}\frac{\partial^2}{\partial x^2}D(x) + \frac{1}{8M^2}\frac{\partial^4}{\partial x^4}A(x)\Big],$$

$$F_2(x) = \frac{1}{4MT}\frac{\partial}{\partial x}D(x) + \frac{1}{4M^2}\frac{\partial^3}{\partial x^3}A(x), \quad F_3(x) = -\frac{1}{2M^2}\frac{\partial^2}{\partial x^2}A(x)$$

Solving the Lindblad equation

University of Stavanger

- Preservation of the continuum properties?

$$\langle\psi_n^{\rm S}|\rho_{\rm S}|\psi_n^{\rm S}\rangle > 0, \forall n, \quad \rho_{\rm S}^\dagger = \rho_{\rm S}, \quad {\rm Tr}[\rho_{\rm S}] = 1$$

required for the probability interpretation

$$\text{Positivity}: \quad \forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C},$$
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

$$\text{Hermiticity}: \quad \rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$$

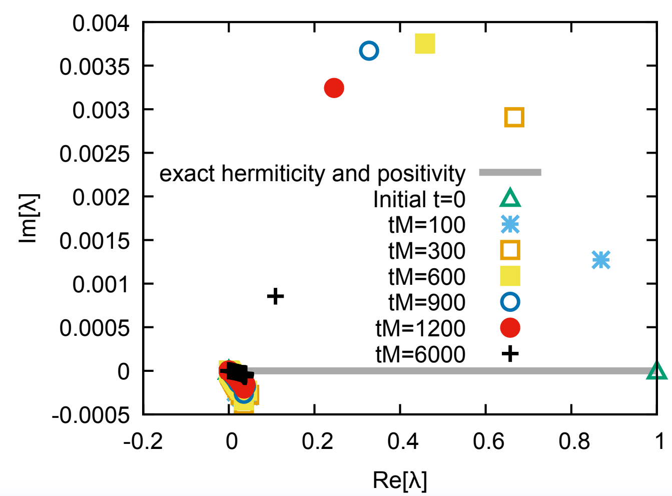
- Starting point: Crank-Nicholson scheme
  (which for non-dissipative dynamics preserves exactly)

$$\rho(x,y,t+\Delta t) = exp[-i\Delta t\mathcal{L}]\rho(x,y,t) \approx \frac{1-\frac{1}{2}i\Delta t\mathcal{L}}{1+\frac{1}{2}i\Delta t\mathcal{L}}\rho(x,y,t)$$

1d Lindblad equation in coord. space

$$\begin{aligned}
\partial_t \rho^{\rm rel}(x,y,t) &= i\Big[\frac{1}{M}\frac{\partial^2}{\partial x^2} - V(x)\Big]\rho^{\rm rel}(x,y,t) - i\Big[\frac{1}{M}\frac{\partial^2}{\partial y^2} - V(y)\Big]\rho^{\rm rel}(x,y,t) \qquad (20)\\
&+\Big[2F_1\Big(\frac{x-y}{2}\Big) - 2F_1(0) + F_1(x) + F_1(y) - 2F_1\Big(\frac{x+y}{2}\Big)\Big]\rho^{\rm rel}(x,y,t)\\
&-\Big[\frac{\partial^2}{\partial x^2}\frac{\frac{\partial^2}{\partial x^2}A(x)}{4M^2} + \frac{\partial^2}{\partial y^2}\frac{\frac{\partial^2}{\partial y^2}A(y)}{4M^2}\Big]\rho^{\rm rel}(x,y,t)\\
&+\Big[2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial x}\frac{\frac{\partial^2}{\partial x^2}A(x)}{M^2}\Big]\frac{\partial}{\partial x}\rho^{\rm rel}(x,y,t)\\
&+\Big[-2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial y}\frac{\frac{\partial^2}{\partial y^2}A(y)}{M^2}\Big]\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
&+\Big[2F_3\Big(\frac{x-y}{2}\Big) + 2F_3\Big(\frac{x+y}{2}\Big)\Big]\frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
&+\Big[F_3(0) + F_3(x)\Big]\frac{\partial^2}{\partial x^2}\rho^{\rm rel}(x,y,t) + \Big[F_3(0) + F_3(y)\Big]\frac{\partial^2}{\partial y^2}\rho^{\rm rel}(x,y,t).
\end{aligned}$$

$$F_1(x) = \Big[D(x) + \frac{1}{4MT}\frac{\partial^2}{\partial x^2}D(x) + \frac{1}{8M^2}\frac{\partial^4}{\partial x^4}A(x)\Big],$$
$$F_2(x) = \frac{1}{4MT}\frac{\partial}{\partial x}D(x) + \frac{1}{4M^2}\frac{\partial^3}{\partial x^3}A(x), \quad F_3(x) = -\frac{1}{2M^2}\frac{\partial^2}{\partial x^2}A(x)$$



Solving the Lindblad equation

University of Stavanger

- Preservation of the continuum properties?

$$\langle\psi_n^{\rm S}|\rho_{\rm S}|\psi_n^{\rm S}\rangle > 0, \forall n, \quad \rho_{\rm S}^\dagger = \rho_{\rm S}, \quad {\rm Tr}[\rho_{\rm S}] = 1$$

required for the probability interpretation

Positivity : $\forall f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots) \in \mathbb{C}$,

$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)^* \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) f(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots) \geq 0$$

Hermiticity : $\rho(\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{x}_2, \boldsymbol{x}_1, t)^* = \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t)$

- Starting point: Crank-Nicholson scheme
(which for non-dissipative dynamics preserves exactly)

$$\rho(x, y, t + \Delta t) = exp[-i\Delta t \mathcal{L}]\rho(x, y, t) \approx \frac{1 - \frac{1}{2}i\Delta t\mathcal{L}}{1 + \frac{1}{2}i\Delta t\mathcal{L}}\rho(x, y, t)$$

- What about the trace conservation? More involved!

required for the physics interpretation

Unit trace :

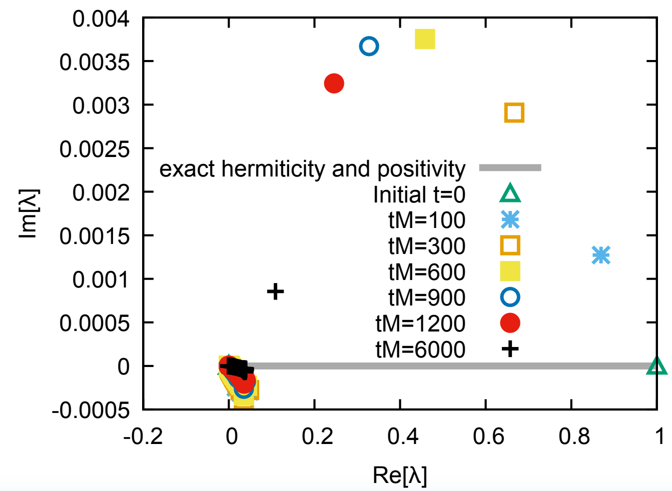
$$\int d^3x_1 d^3x_2 \ldots d^3y_2 d^3y_1 \delta^{(3)}(\boldsymbol{x}_1 - \boldsymbol{y}_1) \ldots \rho(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{y}_2, \boldsymbol{y}_1, t) = 1$$

1d Lindblad equation in coord. space

$$\begin{aligned}
\partial_t \rho^{\rm rel}(x,y,t) &= i\Big[\frac{1}{M}\frac{\partial^2}{\partial x^2} - V(x)\Big]\rho^{\rm rel}(x,y,t) - i\Big[\frac{1}{M}\frac{\partial^2}{\partial y^2} - V(y)\Big]\rho^{\rm rel}(x,y,t) \qquad (20)\\
&+\Big[2F_1\Big(\frac{x-y}{2}\Big) - 2F_1(0) + F_1(x) + F_1(y) - 2F_1\Big(\frac{x+y}{2}\Big)\Big]\rho^{\rm rel}(x,y,t)\\
&-\Big[\frac{\partial^2}{\partial x^2}\frac{\frac{\partial^2}{\partial x^2}A(x)}{4M^2} + \frac{\partial^2}{\partial y^2}\frac{\frac{\partial^2}{\partial y^2}A(y)}{4M^2}\Big]\rho^{\rm rel}(x,y,t)\\
&+\Big[2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial x}\frac{\frac{\partial^2}{\partial x^2}A(x)}{M^2}\Big]\frac{\partial}{\partial x}\rho^{\rm rel}(x,y,t)\\
&+\Big[-2F_2\Big(\frac{x-y}{2}\Big) + 2F_2(x) - 2F_2\Big(\frac{x+y}{2}\Big) - \frac{\partial}{\partial y}\frac{\frac{\partial^2}{\partial y^2}A(y)}{M^2}\Big]\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
&+\Big[2F_3\Big(\frac{x-y}{2}\Big) + 2F_3\Big(\frac{x+y}{2}\Big)\Big]\frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho^{\rm rel}(x,y,t)\\
&+\Big[F_3(0) + F_3(x)\Big]\frac{\partial^2}{\partial x^2}\rho^{\rm rel}(x,y,t) + \Big[F_3(0) + F_3(y)\Big]\frac{\partial^2}{\partial y^2}\rho^{\rm rel}(x,y,t).
\end{aligned}$$

$$F_1(x) = \Big[D(x) + \frac{1}{4MT}\frac{\partial^2}{\partial x^2}D(x) + \frac{1}{8M^2}\frac{\partial^4}{\partial x^4}A(x)\Big],$$

$$F_2(x) = \frac{1}{4MT}\frac{\partial}{\partial x}D(x) + \frac{1}{4M^2}\frac{\partial^3}{\partial x^3}A(x), \quad F_3(x) = -\frac{1}{2M^2}\frac{\partial^2}{\partial x^2}A(x)$$



Trace conservation

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

Trace conservation

University of Stavanger

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\mathrm{rel}}(x,y,t) = 0$$

- Need **summation by parts** difference operators
  (trivial for periodic boundary conditions: usual central derivative)

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \ldots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad \|\mathfrak{u}\|_H^2 = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top (E_N - E_0)\mathfrak{v}$$

$$E_N = \mathrm{diag}[0,\ldots,1] \text{ and } E_0 = \mathrm{diag}[1,\ldots,0]$$

see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

Trace conservation

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

  - Need **summation by parts** difference operators (trivial for periodic boundary conditions: usual central derivative)
  - Product rule broken by finite differences, actually only **reparameterization invariance** needed!

$$\int dx \int dy\, \delta(x-y)\, F_3\left(\frac{x+y}{2}\right) \frac{\partial}{\partial x}\frac{\partial}{\partial y} \rho(x,y,t)$$

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \dots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad ||\mathfrak{u}||_H^2 = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top (E_N - E_0)\mathfrak{v}$$

$$E_N = \mathrm{diag}[0,\dots,1] \text{ and } E_0 = \mathrm{diag}[1,\dots,0]$$

see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

Trace conservation

University of Stavanger

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

  - Need **summation by parts** difference operators
    (trivial for periodic boundary conditions: usual central derivative)
  - Product rule broken by finite differences, actually only **reparameterization invariance** needed!

$$\int dx \int dy\, \delta(x-y)\, F_3(\frac{x+y}{2})\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho(x,y,t)$$
$$= \int dz \int dz'\, \delta(z)\, F_3(\frac{z'}{2})\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right)\rho(z,z',t)$$

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \ldots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad \|\mathfrak{u}\|^2_H = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top(E_N - E_0)\mathfrak{v}$$

$$E_N = \mathrm{diag}[0,\ldots,1] \text{ and } E_0 = \mathrm{diag}[1,\ldots,0]$$

see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

$$z = x - y, \quad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

Trace conservation

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

  - Need **summation by parts** difference operators
    (trivial for periodic boundary conditions: usual central derivative)
  - Product rule broken by finite differences, actually only **reparameterization invariance** needed!

$$\int dx \int dy\, \delta(x-y)\, F_3\Big(\frac{x+y}{2}\Big)\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho(x,y,t)$$
$$= \int dz \int dz'\, \delta(z)\, F_3\Big(\frac{z'}{2}\Big)\Big(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\Big)\rho(z,z',t)$$

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \ldots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad \|\mathfrak{u}\|_H^2 = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top (E_N - E_0)\mathfrak{v}$$

$$E_N = \mathrm{diag}[0,\ldots,1] \text{ and } E_0 = \mathrm{diag}[1,\ldots,0]$$

see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

$z = x - y,$
$z' = x + y$

$$\frac{\partial}{\partial z'} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big)$$

- Solution: a reparameterization invariant SBP finite difference operator
  O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. arXiv:2004.04406

Trace conservation

University of Stavanger

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

  - Need **summation by parts** difference operators
    (trivial for periodic boundary conditions: usual central derivative)
  - Product rule broken by finite differences, actually only **reparameterization invariance** needed!

$$\int dx \int dy\, \delta(x-y)\, F_3\Big(\frac{x+y}{2}\Big)\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho(x,y,t)$$

$$= \int dz \int dz'\, \delta(z)\, F_3\Big(\frac{z'}{2}\Big) \Big(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\Big)\rho(z,z',t)$$

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \dots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad \|\mathfrak{u}\|_H^2 = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top (E_N - E_0)\mathfrak{v}$$

$E_N = \mathrm{diag}[0,\dots,1]$ and $E_0 = \mathrm{diag}[1,\dots,0]$

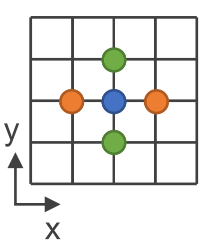
see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

$z = x - y,$
$z' = x + y$

$$\frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

- Solution: a reparameterization invariant SBP finite difference operator

O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. arXiv:2004.04406



$$\mathbb{D}_x u = \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2\Delta}$$

$$\mathbb{D}_y u = \frac{u(x_i, y_{j+1}) - u(x_i, y_{j-1})}{2\Delta}$$

Trace conservation

University of Stavanger

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

- Need **summation by parts** difference operators
  (trivial for periodic boundary conditions: usual central derivative)
- Product rule broken by finite differences, actually only **reparameterization invariance** needed!

$$\int dx \int dy\, \delta(x-y)\, F_3\Big(\frac{x+y}{2}\Big)\, \frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho(x,y,t)$$
$$= \int dz \int dz'\, \delta(z)\, F_3\Big(\frac{z'}{2}\Big) \Big(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\Big)\rho(z,z',t)$$

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \ldots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad \|\mathfrak{u}\|_H^2 = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top (E_N - E_0)\mathfrak{v}$$

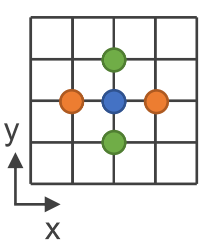
$E_N = \mathrm{diag}[0,\ldots,1]$ and $E_0 = \mathrm{diag}[1,\ldots,0]$

see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

$z = x - y,$
$z' = x + y$

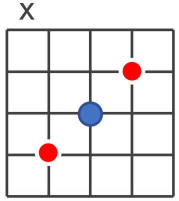
$$\frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

- Solution: a reparameterization invariant SBP finite difference operator
  O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. arXiv:2004.04406



$$\mathbb{D}_x u = \frac{u(x_{i+1}, y_j) - u(x_{i-1}, y_j)}{2\Delta}$$

$$\mathbb{D}_y u = \frac{u(x_i, y_{j+1}) - u(x_i, y_{j-1})}{2\Delta}$$



$$\frac{1}{2}\Big(\mathbb{D}_x + \mathbb{D}_y\Big) u \neq \mathbb{D}_{z'} u = \frac{u(x_{i+1}, y_{j+1}) - u(x_{i-1}, y_{j-1})}{2\Delta}$$

Trace conservation

University of Stavanger

- In the continuum: proof relies on integration by parts and at first sight on product rule

$$T_4 = \int dx \int dy \delta(x-y) \left[i\frac{1}{M}\frac{\partial^2}{\partial x^2} - i\frac{1}{M}\frac{\partial^2}{\partial y^2} \right] \rho^{\rm rel}(x,y,t) = 0$$

  - Need **summation by parts** difference operators (trivial for periodic boundary conditions: usual central derivative)
  - Product rule broken by finite differences, actually only **reparameterization invariance** needed!

$$\int dx \int dy\, \delta(x-y)\, F_3\!\left(\frac{x+y}{2}\right) \frac{\partial}{\partial x}\frac{\partial}{\partial y}\rho(x,y,t)$$
$$= \int dz \int dz'\, \delta(z)\, F_3\!\left(\frac{z'}{2}\right) \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z'^2}\right)\rho(z,z',t)$$

Summation by parts

Function: $\mathfrak{u} = (u(x_0), u(x_1), \ldots)$

Integration scheme/BC: $(\mathfrak{u},\mathfrak{v})_H = \mathfrak{u}^\top H \mathfrak{v}, \quad \|\mathfrak{u}\|^2_H = (\mathfrak{u},\mathfrak{u})_H$

SBP property: $D\mathfrak{u} \approx \mathfrak{u}_x$

$$(\mathfrak{v}, D\mathfrak{u})_H = -(\mathfrak{u}, D\mathfrak{v})_H + \mathfrak{u}^\top (E_N - E_0)\mathfrak{v}$$

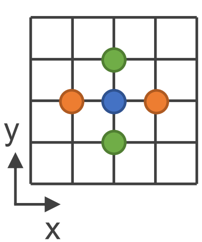
$$E_N = {\rm diag}[0,\ldots,1] \text{ and } E_0 = {\rm diag}[1,\ldots,0]$$

see e.g. D. C. D. R. Fernández, et.al., Computers & Fluids 95 (2014) 171–196.

$$z = x - y, \quad z' = x + y \qquad \frac{\partial}{\partial z'} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

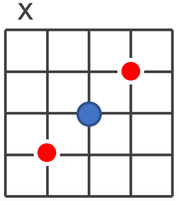
- Solution: a reparameterization invariant SBP finite difference operator

O. Ålund, Y. Akamatsu, F. Laurén, T.Miura, J. Nordström, A.R. arXiv:2004.04406



$$\mathbb{D}_x u = \frac{u(x_{i+1},y_j) - u(x_{i-1},y_j)}{2\Delta}$$

$$\mathbb{D}_y u = \frac{u(x_i,y_{j+1}) - u(x_i,y_{j-1})}{2\Delta}$$

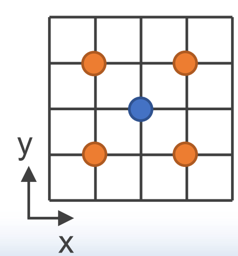


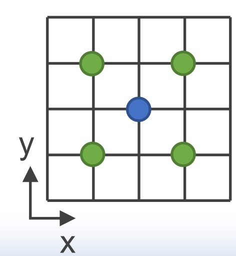
$$\frac{1}{2}\left(\mathbb{D}_x + \mathbb{D}_y\right) u \neq \mathbb{D}_{z'} u = \frac{u(x_{i+1},y_{j+1}) - u(x_{i-1},y_{j-1})}{2\Delta}$$

$$\mathbb{D}^{\rm RN}_x u = \frac{u(x_{i+1},y_{j+1}) - u(x_{i-1},y_{j+1}) + u(x_{i+1},y_{j-1}) - u(x_{i-1},y_{j-1})}{4\Delta}$$

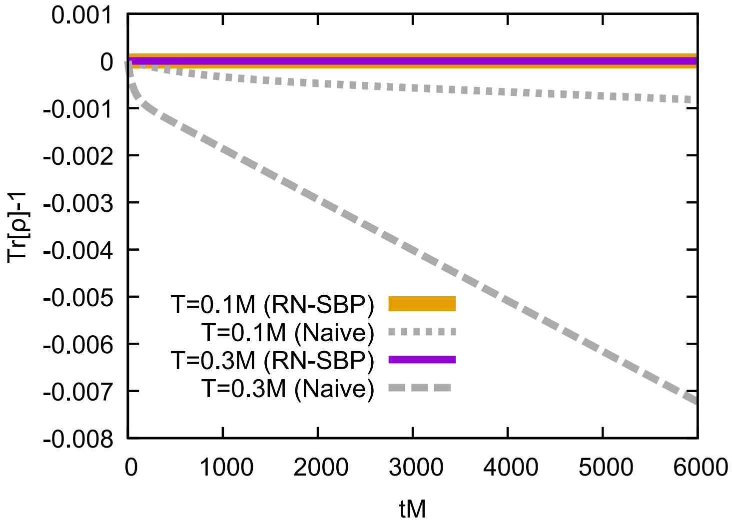
$$\mathbb{D}^{\rm RN}_y u = \frac{u(x_{i+1},y_{j+1}) - u(x_{i+1},y_{j-1}) + u(x_{i-1},y_{j+1}) - u(x_{i-1},y_{j-1})}{4\Delta}$$

$$\frac{1}{2}\left(\mathbb{D}^{\rm RN}_x + \mathbb{D}^{\rm RN}_y\right) u = \mathbb{D}_{z'} u$$

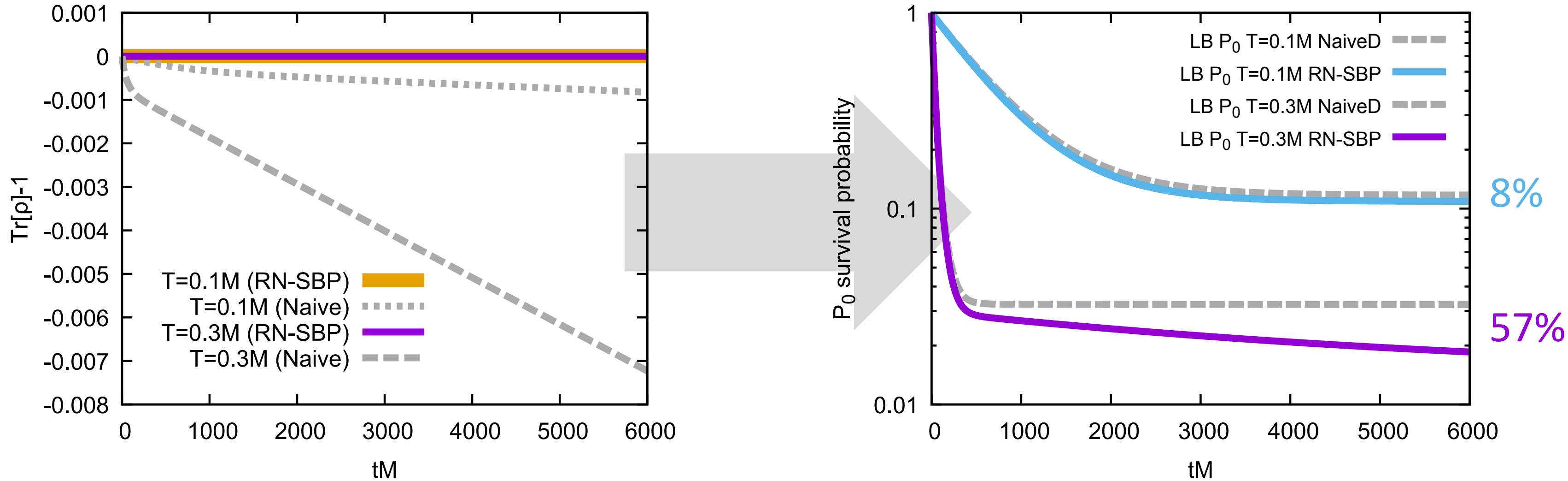




Recent simulation results

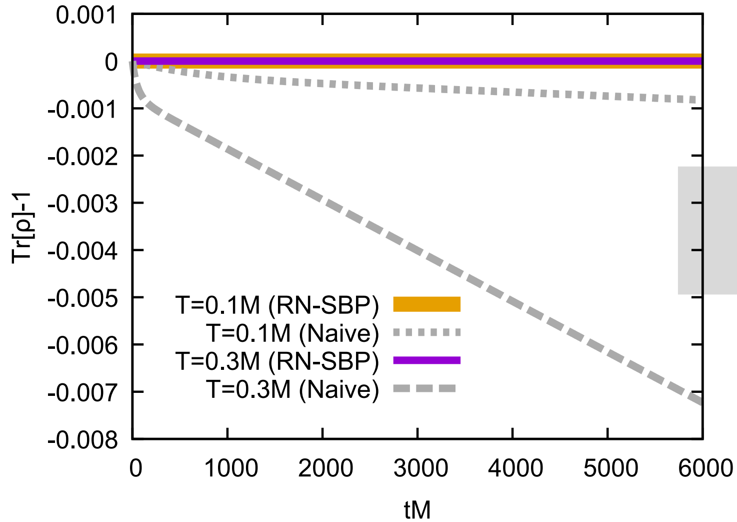


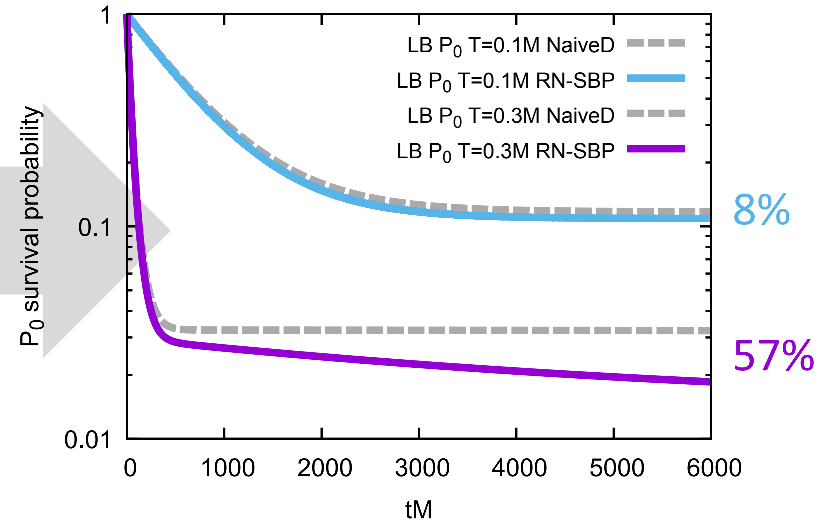
Recent simulation results



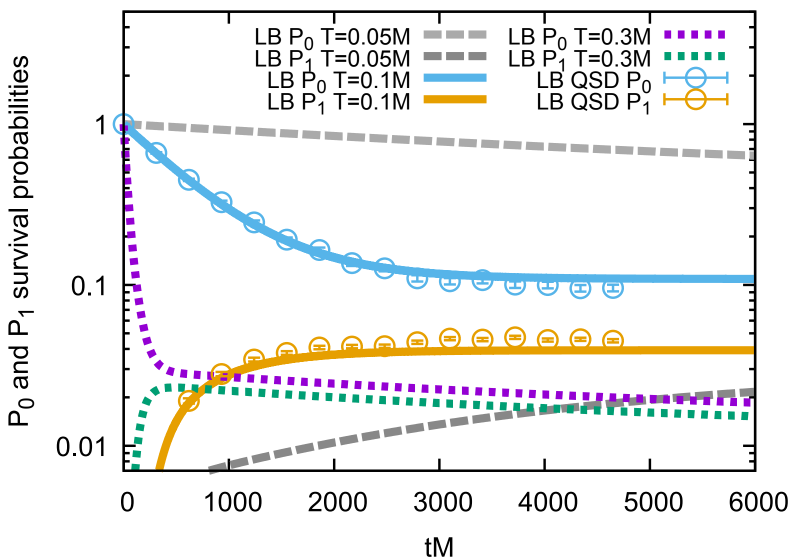
- Tiny deviations from overall trace conservation can translate into significant survival mismatch

Recent simulation results





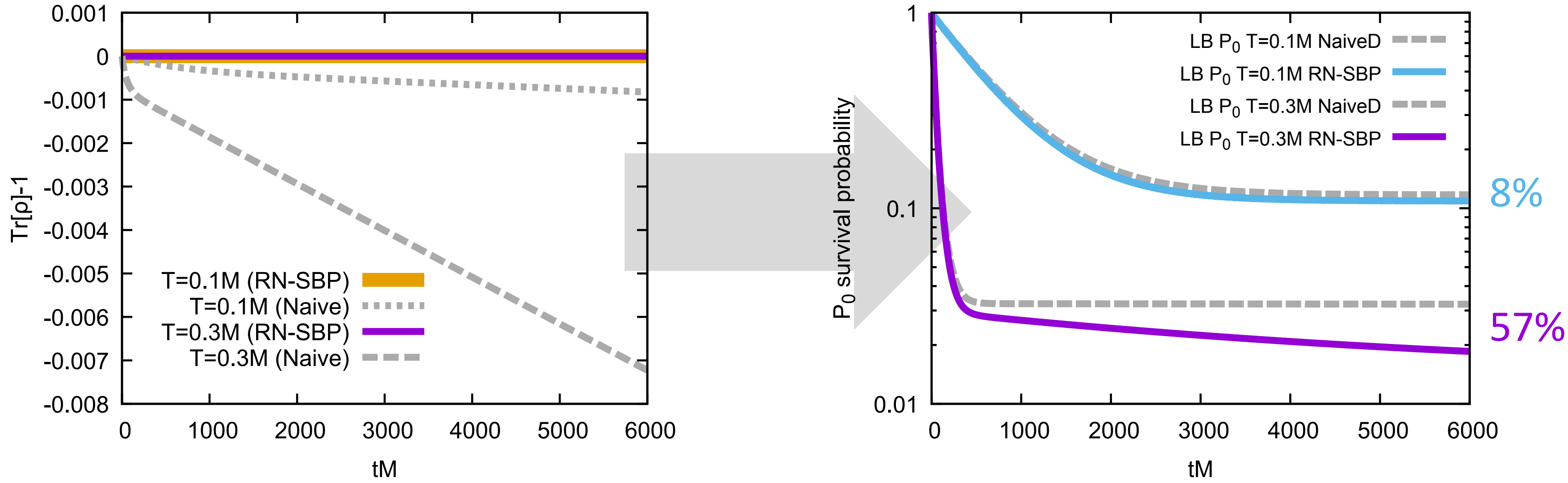
- Tiny deviations from overall trace conservation can translate into significant survival mismatch



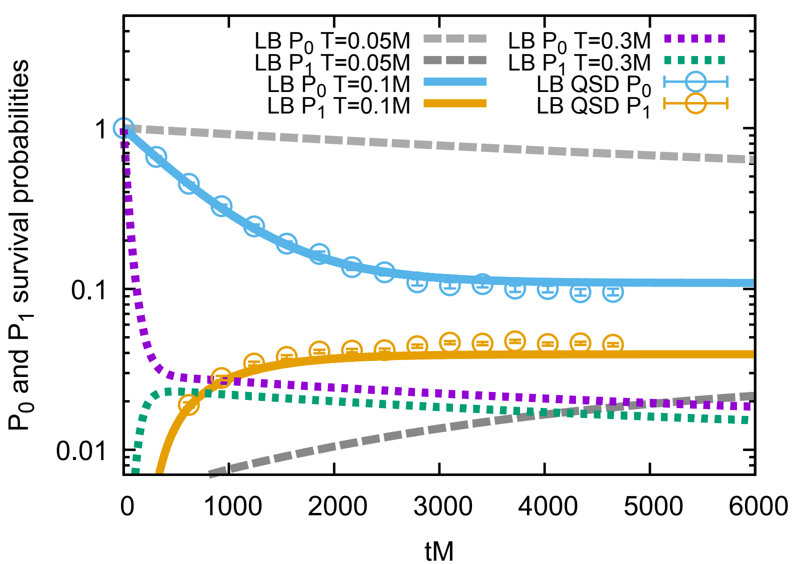
- Direct solution of master equation does not require additional approximations such as employed in stochastic unravelling (QSD)

Recent simulation results

University of Stavanger



- Tiny deviations from overall trace conservation can translate into significant survival mismatch



- Direct solution of master equation does not require additional approximations such as employed in stochastic unravelling (QSD)
- Curse of dimensionality comes back to haunt us in 3d: density matrix is a 6d object.

University of Stavanger

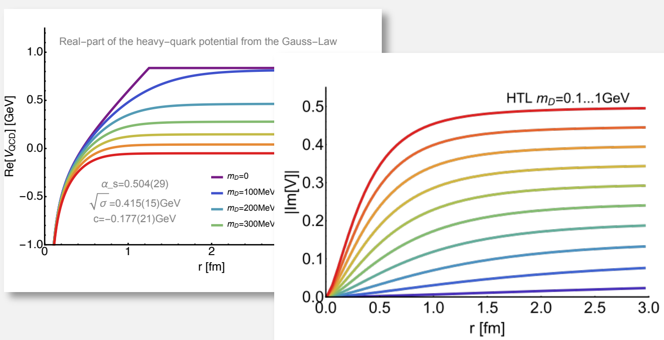
Challenge II:
The heavy-quark potential in the classical limit

A puzzle in the literature

University of Stavanger

Quantum computation

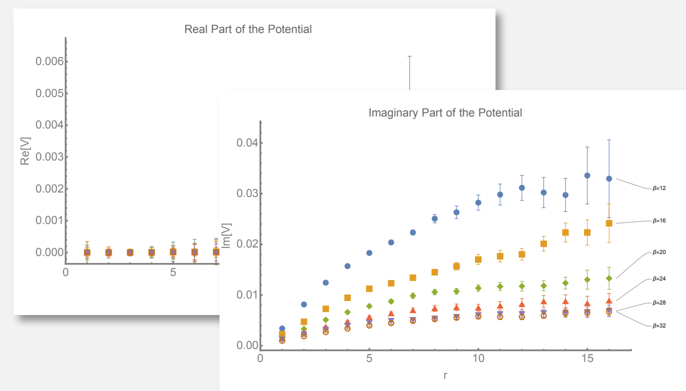
- Static quark interactions can be described by a potential with a real and imaginary part



- Manifestation of weakend quark binding and kicks with hot environment partons.

Classical computation

- Interquark potential shows only imaginary part but no real part.



see e.g. Laine, Philipsen, Tassler, JHEP 09 (2007) 066

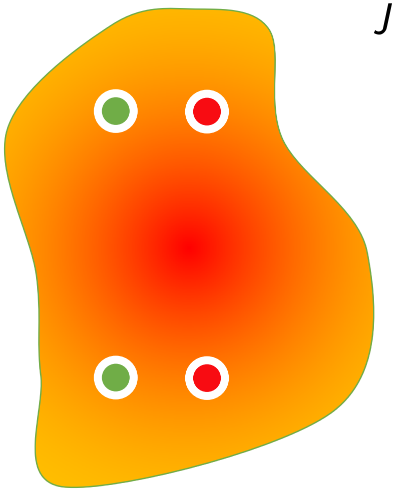
- Obviously, no confinement in a classical theory but attractive interactions possible in a non-linear Maxwell-like theory (YM).
- Confusing since classical Maxwell theory predicts Debye screened potential

How do static QCD charges interact?

University of Stavanger

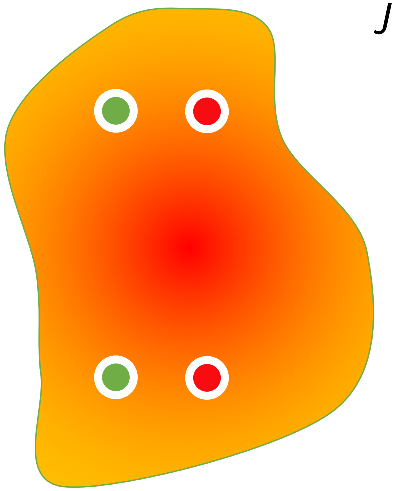
- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



How do static QCD charges interact?

- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1



$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

QCD path integral: introduce charges as Wilson line correlators

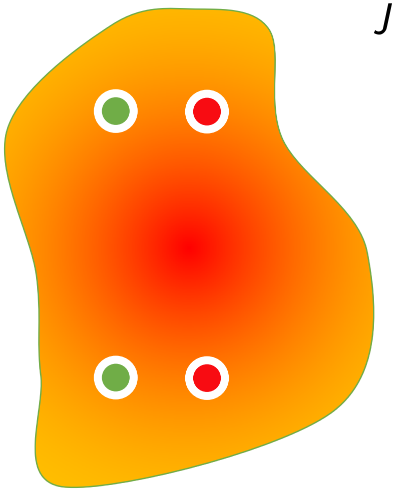
$$\int \mathcal{D}A\, exp\Big[i\int d^4z Tr\Big\{-\frac{1}{2}F^{\mu\nu}(z)F_{\mu\nu}(z) - A^0(z)j_0(z)\Big\}\Big]$$

How do static QCD charges interact?

University of Stavanger

- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



QCD path integral: introduce charges as Wilson line correlators

$$\int \mathcal{D}A\, exp\Big[i\int d^4z Tr\Big\{-\frac{1}{2}F^{\mu\nu}(z)F_{\mu\nu}(z) - A^0(z)j_0(z)\Big\}\Big]$$

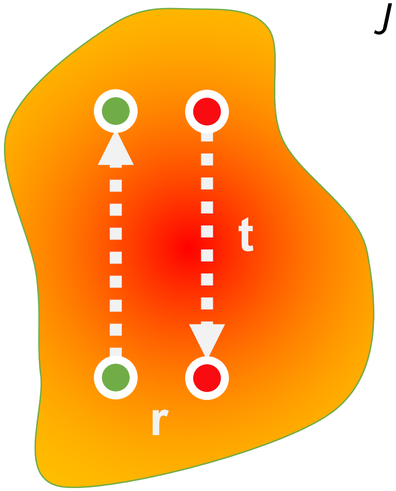
$$= \int \mathcal{D}A\; exp[i\int_{t_i}^{t_f} dtA^0(\mathbf{x}_0,t)]exp[i\int_{t_f}^{t_i} dtA^0(\mathbf{x}_1,t)]\; exp\Big[iS_{YM}\Big]$$

How do static QCD charges interact?

University of Stavanger

- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



QCD path integral: introduce charges as Wilson line correlators

$$\int \mathcal{D}A\, exp\Big[i \int d^4z Tr\Big\{ -\frac{1}{2}F^{\mu\nu}(z)F_{\mu\nu}(z) - A^0(z)j_0(z)\Big\}\Big]$$

$$= \int \mathcal{D}A\, exp[i \int_{t_i}^{t_f} dt A^0(\mathbf{x}_0,t)] exp[i \int_{t_f}^{t_i} dt A^0(\mathbf{x}_1,t)]\ exp\Big[iS_{YM}\Big]$$

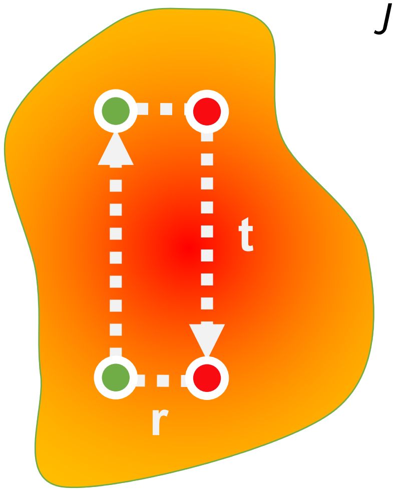
$$= \langle W_{||}\rangle(r = |\mathbf{x}_0 - \mathbf{x}_1|, t = t_f - t_i)$$

How do static QCD charges interact?

University of Stavanger

- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



QCD path integral: introduce charges as Wilson line correlators

$$\int \mathcal{D}A\, exp\Big[i \int d^4 z Tr\Big\{ -\frac{1}{2}F^{\mu\nu}(z)F_{\mu\nu}(z) - A^0(z)j_0(z)\Big\}\Big]$$

$$= \int \mathcal{D}A\, exp[i \int_{t_i}^{t_f} dt A^0(\mathbf{x}_0,t)] exp[i \int_{t_f}^{t_i} dt A^0(\mathbf{x}_1,t)]\; exp\Big[iS_{YM}\Big]$$

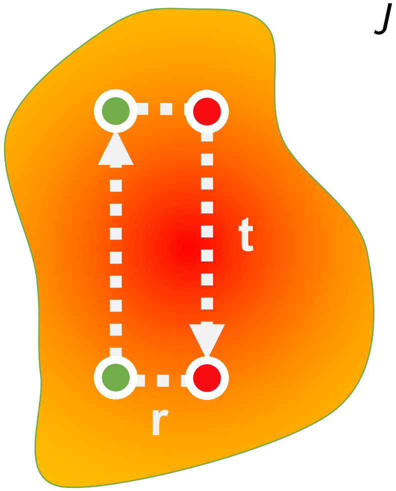
$$= \langle W_{||}\rangle(r = |\mathbf{x}_0 - \mathbf{x}_1|, t = t_f - t_i)$$

How do static QCD charges interact?

University of Stavanger

- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



QCD path integral: introduce charges as Wilson line correlators

$$\int \mathcal{D}A\, exp\Big[i\int d^4z Tr\Big\{-\frac{1}{2}F^{\mu\nu}(z)F_{\mu\nu}(z) - A^0(z)j_0(z)\Big\}\Big]$$

$$= \int \mathcal{D}A\, exp[i\int_{t_i}^{t_f} dtA^0(\mathbf{x}_0,t)]exp[i\int_{t_f}^{t_i} dtA^0(\mathbf{x}_1,t)]\; exp\Big[iS_{YM}\Big]$$

$$= \langle W_{||}\rangle(r=|\mathbf{x}_0-\mathbf{x}_1|, t=t_f-t_i)$$

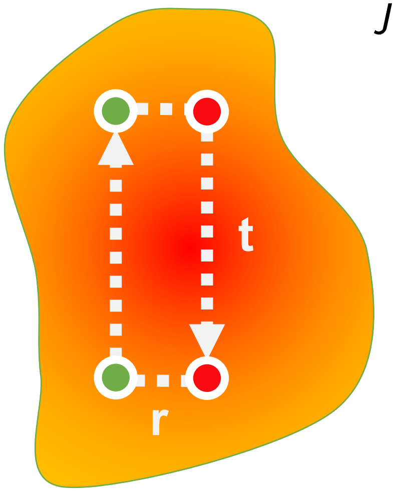
- Evaluating the Wilson correlator in the path integral amounts to reweighting from a theory without static sources to a theory with static sources.

How do static QCD charges interact?

University of Stavanger

- Consider a color neutral pair of static charges (e.g. red-antired) at position x_0,x_1

$$j^0(\mathbf{x}) = M\Big(\delta^{(3)}(\mathbf{x}-\mathbf{x}_0) - \delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\Big) \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



QCD path integral: introduce charges as Wilson line correlators

$$\int \mathcal{D}A\, exp\Big[i\int d^4z Tr\Big\{-\frac{1}{2}F^{\mu\nu}(z)F_{\mu\nu}(z) - A^0(z)j_0(z)\Big\}\Big]$$

$$= \int \mathcal{D}A\; exp[i\int_{t_i}^{t_f} dt A^0(\mathbf{x}_0,t)]exp[i\int_{t_f}^{t_i} dt A^0(\mathbf{x}_1,t)]\; exp\Big[iS_{YM}\Big]$$

$$=\langle W_{||}\rangle(r=|\mathbf{x}_0-\mathbf{x}_1|, t=t_f-t_i)$$

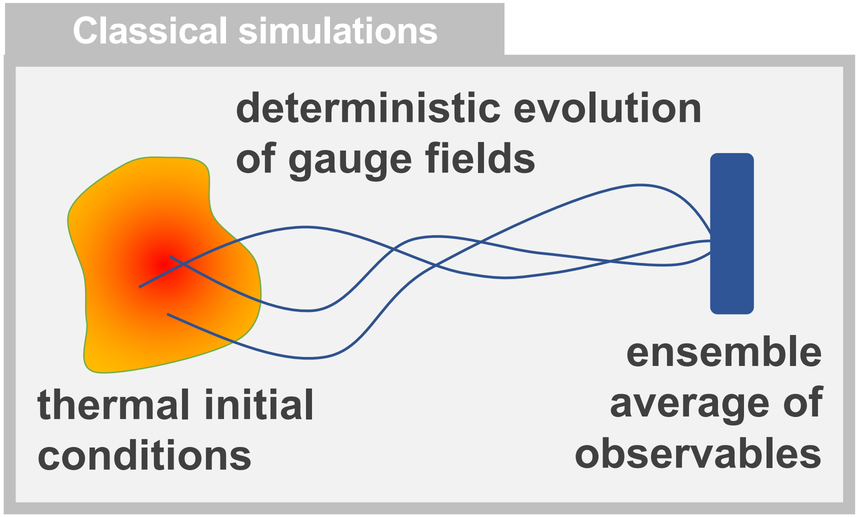
- Evaluating the Wilson correlator in the path integral amounts to reweighting from a theory without static sources to a theory with static sources.

- If Wilson loop follows a Schrödinger-like equation: definition of potential possible

$$i\partial_t\langle W(r,t)\rangle = \Big(ReV(r) - iImV(r)\Big)\langle W(r,t)\rangle$$

In the classical limit

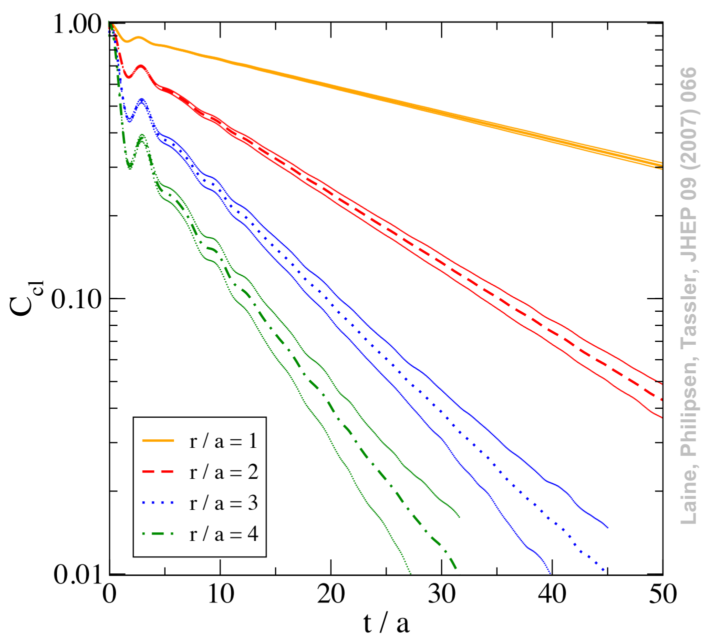
- Solving the Yang-Mills equation of motion for different stochastic initial conditions.

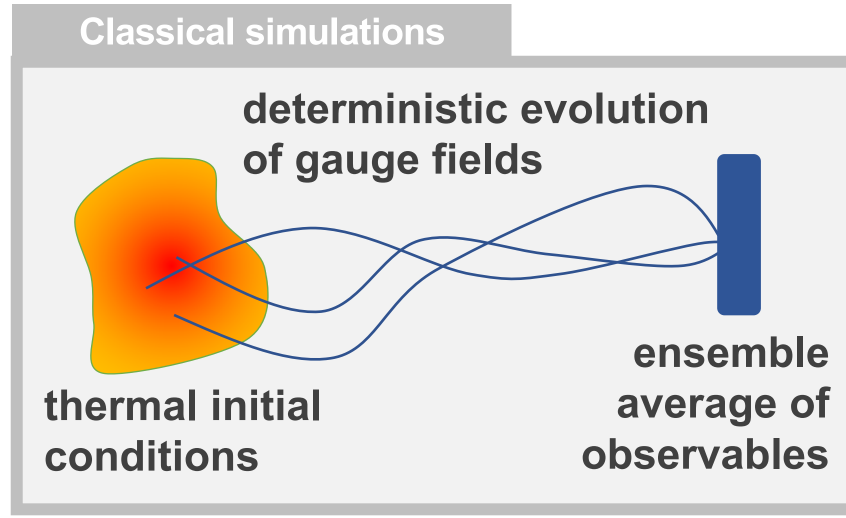


In the classical limit

- Solving the Yang-Mills equation of motion for different stochastic initial conditions.

- In the literature: compute Wilson loop in a thermal ensemble of gauge fields

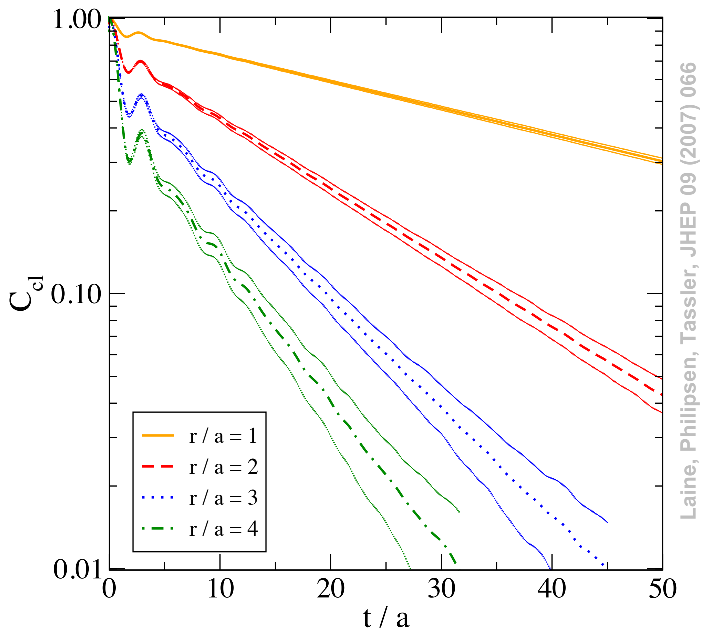


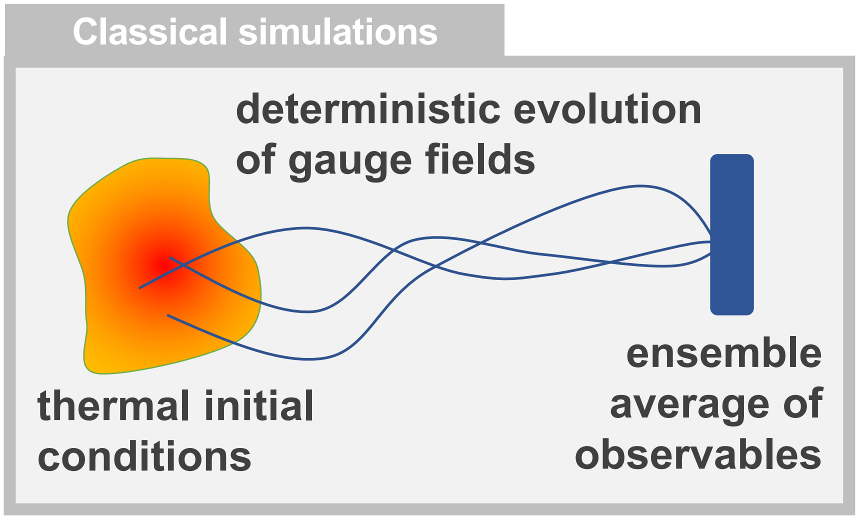


In the classical limit

- Solving the Yang-Mills equation of motion for different stochastic initial conditions.

- In the literature: compute Wilson loop in a thermal ensemble of gauge fields





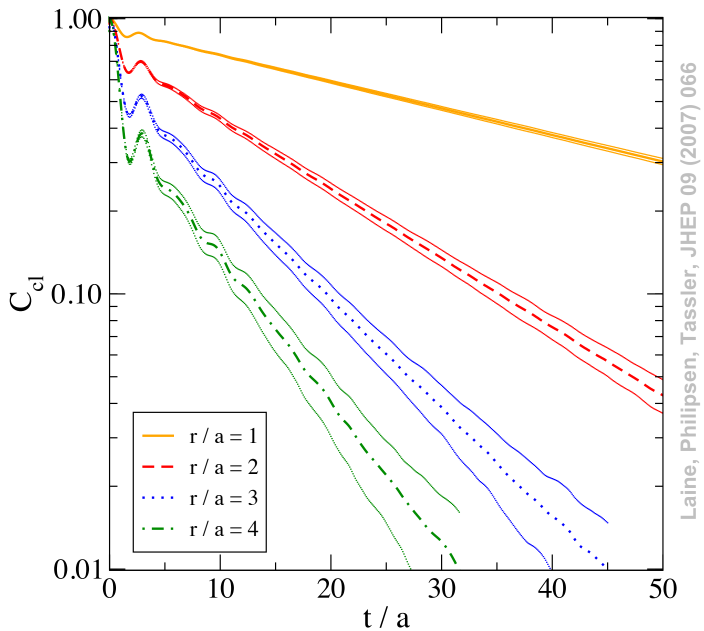
$$\langle W_\square \rangle(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-tImV^{(cl)}(r)]$$

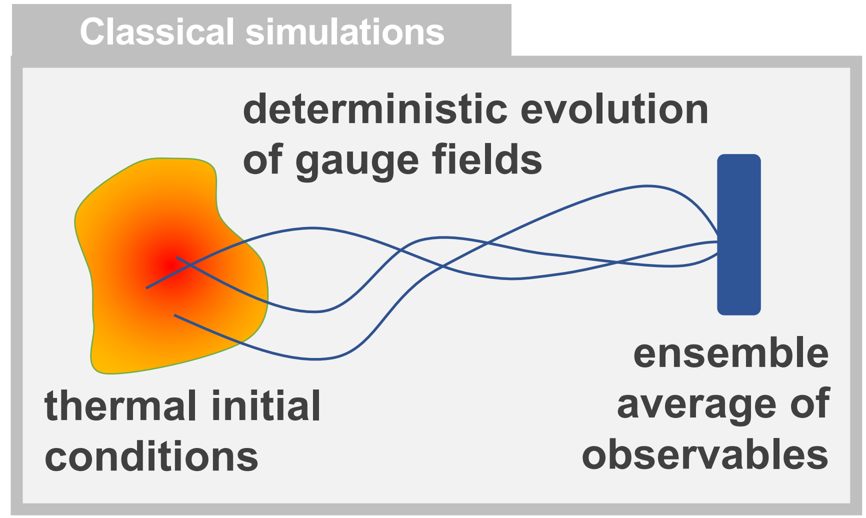
In the classical limit

University of Stavanger

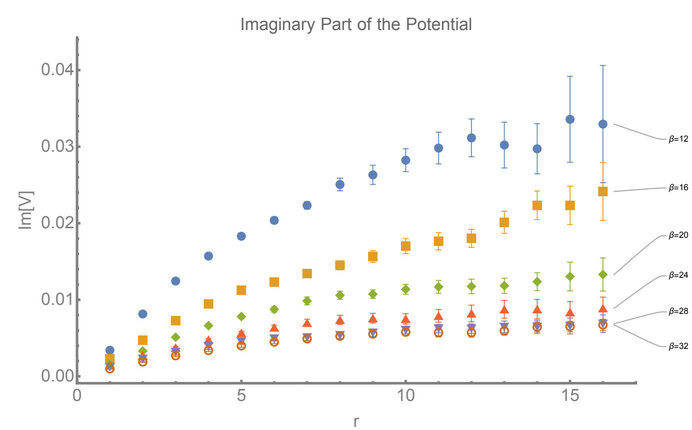
- Solving the Yang-Mills equation of motion for different stochastic initial conditions.

- In the literature: compute Wilson loop in a thermal ensemble of gauge fields





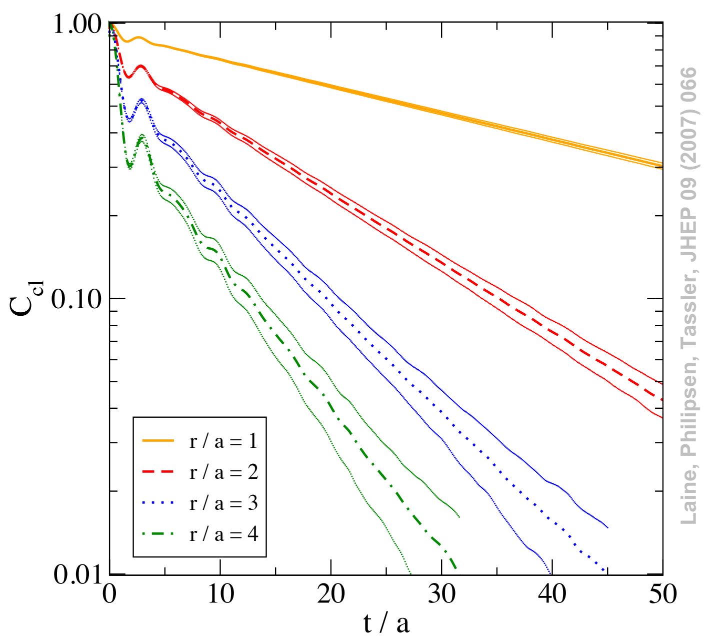
$$\langle W_\square \rangle(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-tImV^{(cl)}(r)]$$

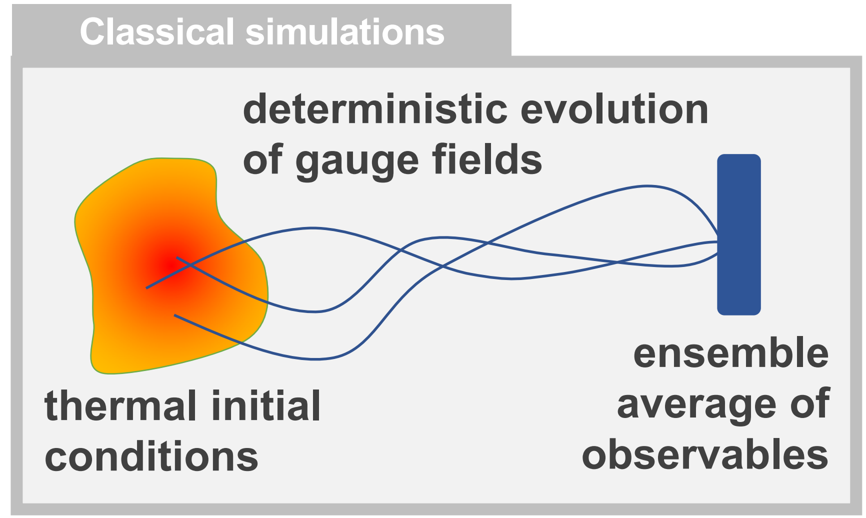


In the classical limit

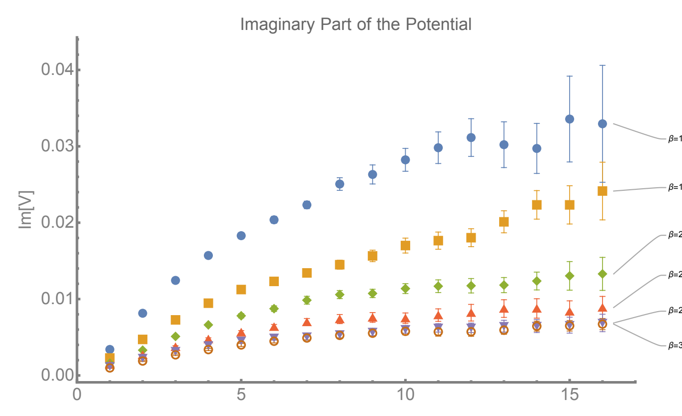
- Solving the Yang-Mills equation of motion for different stochastic initial conditions.

- In the literature: compute Wilson loop in a thermal ensemble of gauge fields





$$\langle W_\square \rangle(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-tImV^{(cl)}(r)]$$



- ImV increases with distance & temperature, qualitatively similar to quantum result

Taking a closer look

- Where do the static sources enter in the classical theory? The Gauss-Law !

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \Big[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t) E_i(\mathbf{x}-i,t) U^{\dagger}_{-i}(\mathbf{x},t) \Big] = 0$$

Taking a closer look

University of Stavanger

- Where do the static sources enter in the classical theory? The Gauss-Law !

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \Big[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t) E_i(\mathbf{x}-i,t) U^{\dagger}_{-i}(\mathbf{x},t) \Big] - \rho(x;x_0,x_1) = 0$$

Taking a closer look

- Where do the static sources enter in the classical theory? The Gauss-Law !

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \Big[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t) E_i(\mathbf{x}-i,t) U^{\dagger}_{-i}(\mathbf{x},t) \Big] - \rho(x;x_0,x_1) = 0$$

- Gauss-Law plays a central role in drawing the stochastic initial conditions

Initial set of U and E:

$$\exp[-\beta_G H_{cl}] \prod_{\mathbf{x}} \delta(G(\mathbf{x}))$$

- Since $P(E)=\exp[-E^2]$ draw from Gaussian
- Project onto G=0 surface
- Mix U and E via e.o.m

Taking a closer look

University of Stavanger

- Where do the static sources enter in the classical theory? The Gauss-Law !

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \left[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t) E_i(\mathbf{x}-i,t) U^{\dagger}_{-i}(\mathbf{x},t) \right] - \rho(x;x_0,x_1) = 0$$

- Gauss-Law plays a central role in drawing the stochastic initial conditions

Initial set of U and E:

$$\exp[-\beta_G H_{cl}] \prod_{\mathbf{x}} \delta(G(\mathbf{x}))$$

- Since $P(E)=\exp[-E^2]$ draw from Gaussian
- Project onto G=0 surface
- Mix U and E via e.o.m

Evolve e.o.m.

- Symplectic leap-frog scheme for E and U
- Good energy and Gauss-Law conservation

Taking a closer look

University of Stavanger

- Where do the static sources enter in the classical theory? The Gauss-Law !

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \left[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t) E_i(\mathbf{x}-i,t) U^{\dagger}_{-i}(\mathbf{x},t) \right] - \rho(x;x_0,x_1) = 0$$

- Gauss-Law plays a central role in drawing the stochastic initial conditions

Initial set of U and E:

$$\exp[-\beta_G H_{cl}] \prod_{\mathbf{x}} \delta(G(\mathbf{x}))$$

- Since $P(E)=\exp[-E^2]$ draw from Gaussian
- Project onto G=0 surface
- Mix U and E via e.o.m

Evolve e.o.m.

- Symplectic leap-frog scheme for E and U
- Good energy and Gauss-Law conservation

Compute the Wilson correlator

Taking a closer look

University of Stavanger

- Where do the static sources enter in the classical theory? The Gauss-Law !

$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}} \left[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t) E_i(\mathbf{x}-i,t) U^{\dagger}_{-i}(\mathbf{x},t) \right] - \rho(x;x_0,x_1) = 0$$

- Gauss-Law plays a central role in drawing the stochastic initial conditions

Initial set of U and E:

$$\exp[-\beta_G H_{cl}] \prod_{\mathbf{x}} \delta(G(\mathbf{x}))$$

- Since $P(E)=\exp[-E^2]$ draw from Gaussian
- Project onto G=0 surface
- Mix U and E via e.o.m

Evolve e.o.m.

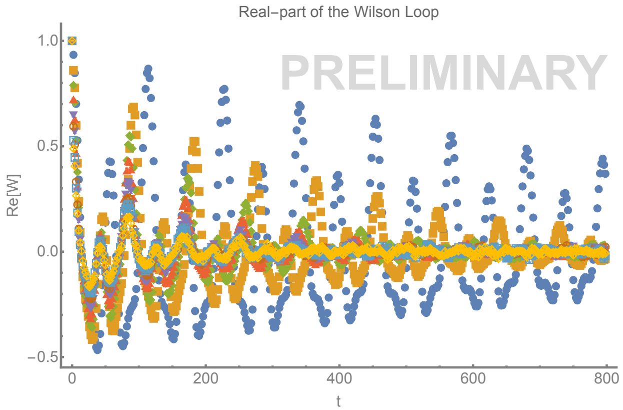
- Symplectic leap-frog scheme for E and U
- Good energy and Gauss-Law conservation

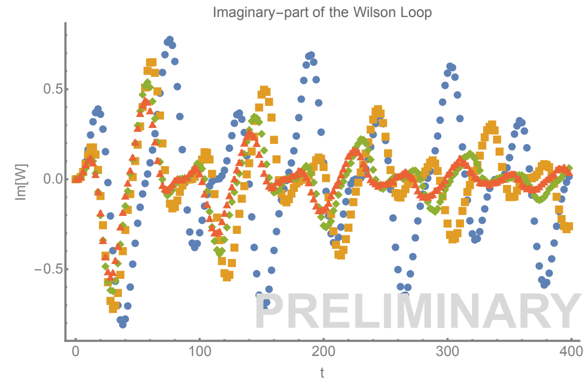
Compute the Wilson correlator

- Evaluating the Wilson loop in the classical simulation does NOT introduce the sources but instead we need to enforce a modified Gauss-Law by hand.

Classic potential in the presence of sources

- With appropriate Gauss-Law: Wilson loop shows much more intricate behavior

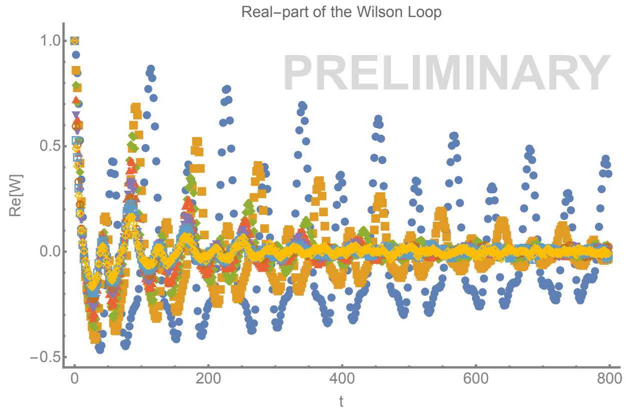


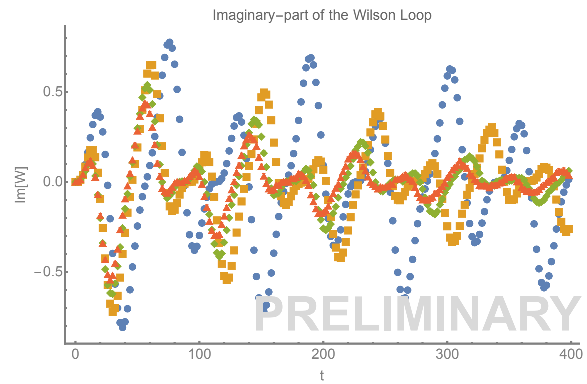


N=32
$a_s=1$
$a_t=0.1$
$\beta=1/a_sT=32$

Classic potential in the presence of sources

- With appropriate Gauss-Law: Wilson loop shows much more intricate behavior





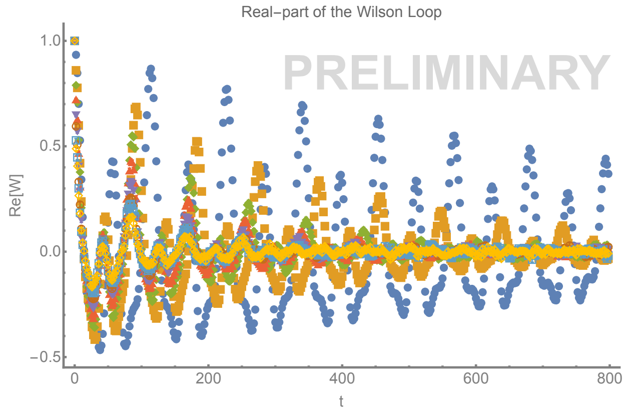
N=32
$a_s=1$
$a_t=0.1$
$\beta=1/a_sT=32$

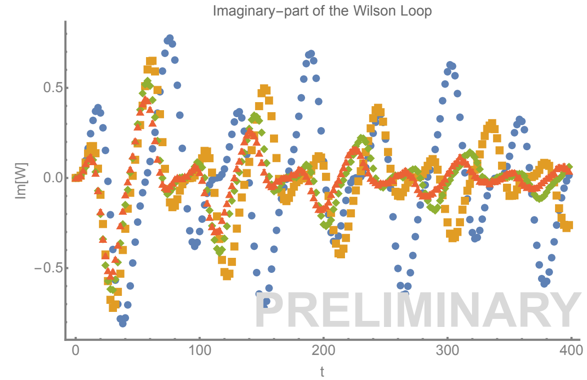
$$\langle W_\square \rangle_{\text{static source}}(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-itReV^{(cl)}(r)]\, exp[-tImV^{(cl)}(r)]$$

Classic potential in the presence of sources

University of Stavanger

- With appropriate Gauss-Law: Wilson loop shows much more intricate behavior

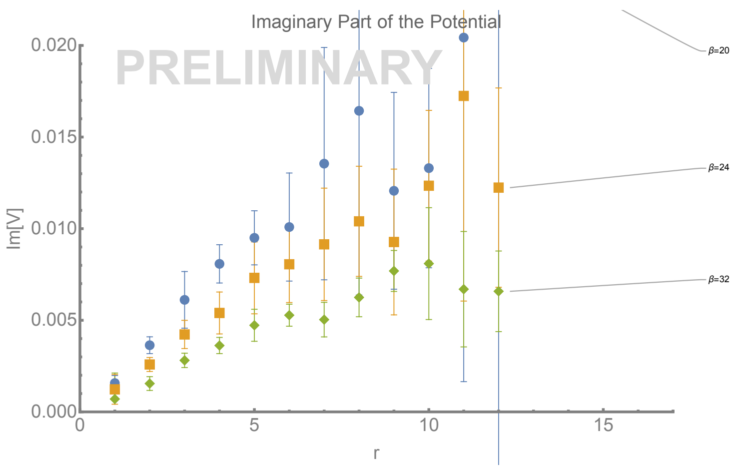




N=32
a_s=1
a_t=0.1
$\beta=1/a_sT=32$

$$\langle W_\square \rangle_{\text{static source}}(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-itReV^{(cl)}(r)]\, exp[-tImV^{(cl)}(r)]$$

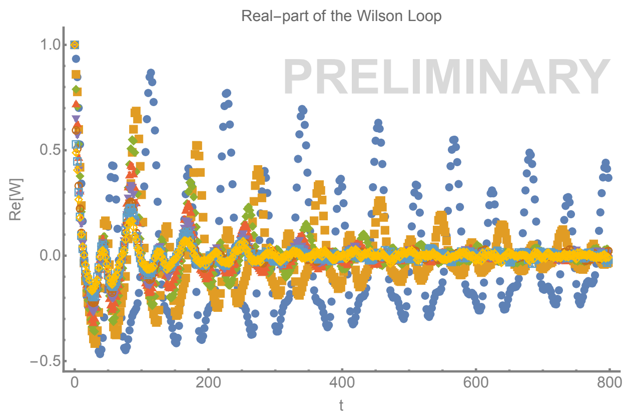
- Oscillatory part leads to a real-part, damping to an imaginary part in the potential

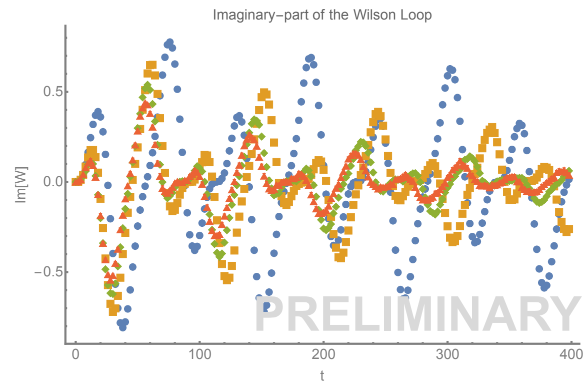


Classic potential in the presence of sources

University of Stavanger

- With appropriate Gauss-Law: Wilson loop shows much more intricate behavior

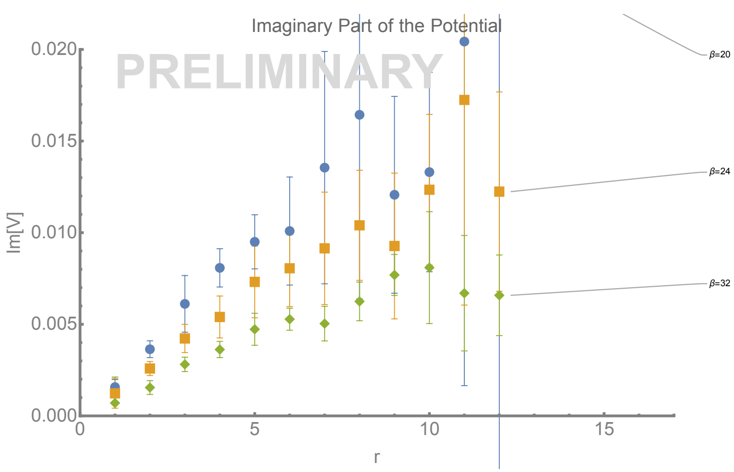


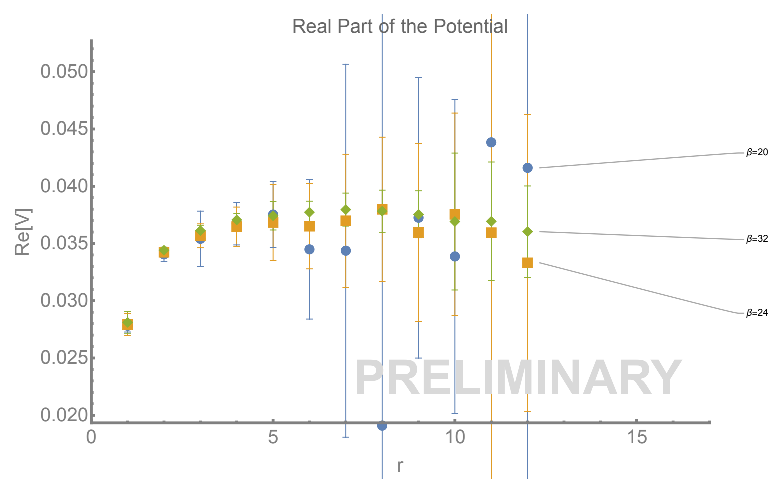


N=32
a_s=1
a_t=0.1
$\beta=1/a_sT=32$

$$\langle W_\square \rangle_{\text{static source}}(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-itReV^{(cl)}(r)]\, exp[-tImV^{(cl)}(r)]$$

- Oscillatory part leads to a real-part, damping to an imaginary part in the potential

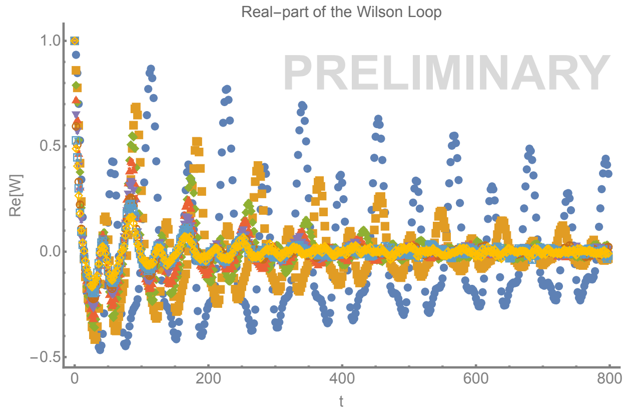


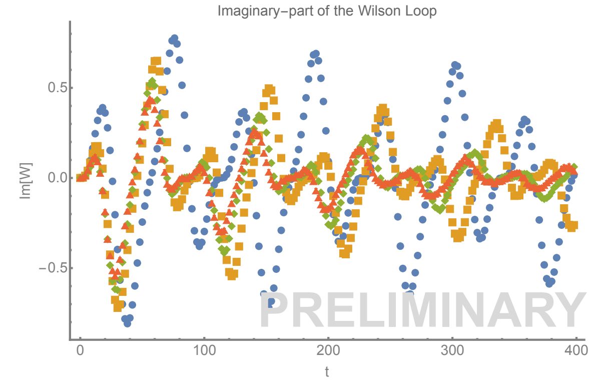


Classic potential in the presence of sources

University of Stavanger

- With appropriate Gauss-Law: Wilson loop shows much more intricate behavior

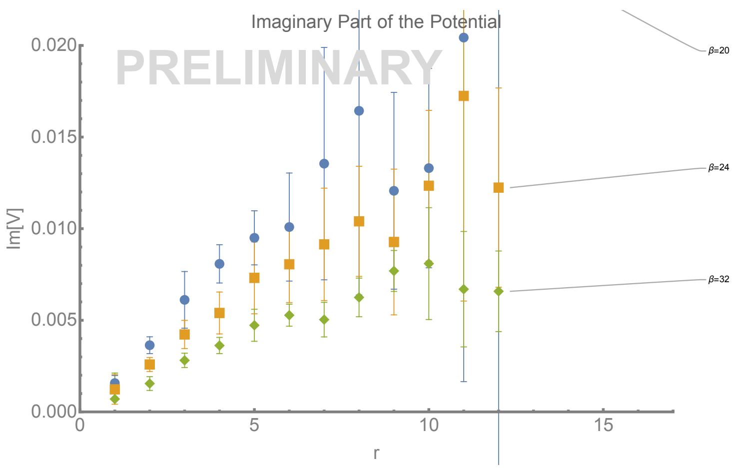


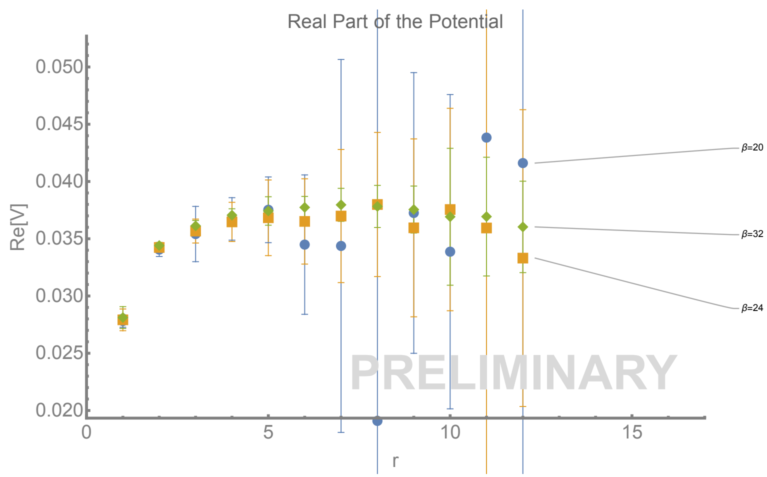


N=32
a_s=1
a_t=0.1
β=1/a_sT=32

$$\langle W_\square \rangle_{\text{static source}}(r,t) \approx exp[-itV^{(cl)}(r)] = exp[-itReV^{(cl)}(r)]\, exp[-tImV^{(cl)}(r)]$$

- Oscillatory part leads to a real-part, damping to an imaginary part in the potential





- Screened potential consistent with expectation from classical Debye theory

Summary

- Quarkonium is a fascinating laboratory for the strong interactions
- Steady progress in understanding Quarkonium dynamics
  - Access to real-time dynamics via **open-quantum-systems** framework
  - New numerical techniques to simulate 2-body master equations accurately: **reparametrization-neutral** summation-by-parts **finite difference operator**
  - Correct treatment of **static sources** in classical Yang-Mills simulations reveals the presence of a **screened real-part** of the potential
- A lot of things remain to be done, e.g:
  - Push the master equation simulation to full three-dimensions
  - Develop real-time simulations for finite mass heavy quarks (lattice NRQCD)
  - Improve the extraction of spectral functions from lattice QCD (Gaurang's talk)

Summary

- Quarkonium is a fascinating laboratory for the strong interactions
- Steady progress in understanding Quarkonium dynamics
  - Access to real-time dynamics via **open-quantum-systems** framework
  - New numerical techniques to simulate 2-body master equations accurately: **reparametrization-neutral** summation-by-parts **finite difference operator**
  - Correct treatment of **static sources** in classical Yang-Mills simulations reveals the presence of a **screened real-part** of the potential
- A lot of things remain to be done, e.g:
  - Push the master equation simulation to full three-dimensions
  - Develop real-time simulations for finite mass heavy quarks (lattice NRQCD)
  - Improve the extraction of spectral functions from lattice QCD (Gaurang's talk)

Thank you for your attention

Potential from spectral functions

- Potential emerges at late times: coarse graining of multiple gluon exchanges

$$V(R) = \lim_{t\to\infty} \frac{i\partial_t W_\square(R,t)}{W_\square(R,t)} \qquad\qquad W_\square(R,t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \rho_\square(R,\omega)$$

Potential from spectral functions

University of Stavanger

- Potential emerges at late times: coarse graining of multiple gluon exchanges

$$V(R) = \lim_{t\to\infty} \frac{i\partial_t W_\square(R,t)}{W_\square(R,t)}$$

$$W_\square(R,t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \rho_\square(R,\omega)$$

Spectral Decomposition

$$V^{QCD}(R) = \lim_{t\to\infty} \frac{\int_{-\infty}^{\infty} d\omega\, \omega\, e^{-i\omega t}\, \rho_\square(R,\omega)}{\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, \rho_\square(R,\omega)}$$

A.R., T.Hatsuda & S.Sasaki PoS LAT2009 (2009) 162

Potential from spectral functions

- Potential emerges at late times: coarse graining of multiple gluon exchanges

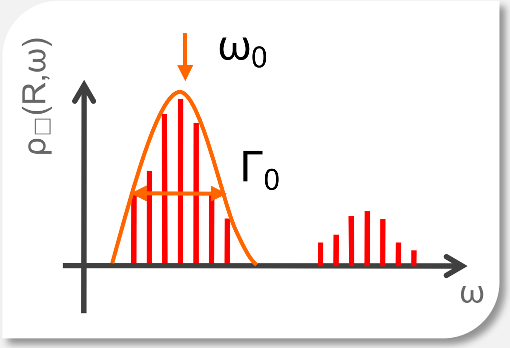
$$V(R) = \lim_{t\to\infty} \frac{i\partial_t W_\square(R,t)}{W_\square(R,t)}$$

$$W_\square(R,t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \rho_\square(R,\omega)$$

Spectral Decomposition

$$V^{QCD}(R) = \lim_{t\to\infty} \frac{\int_{-\infty}^{\infty} d\omega\, \omega\, e^{-i\omega t} \rho_\square(R,\omega)}{\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \rho_\square(R,\omega)}$$

A.R., T.Hatsuda & S.Sasaki PoS LAT2009 (2009) 162



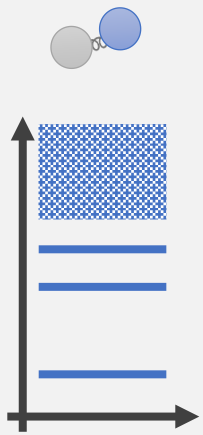
well defined V(R) if low lying Breit-Wigner present in Wilson loop spectral function

For technical details see Y.B., A.R. PRD86 (2012) 051503

$$V(R) = \omega_0(R) - i\Gamma_0(R)$$

Building intuition on Quarkonium

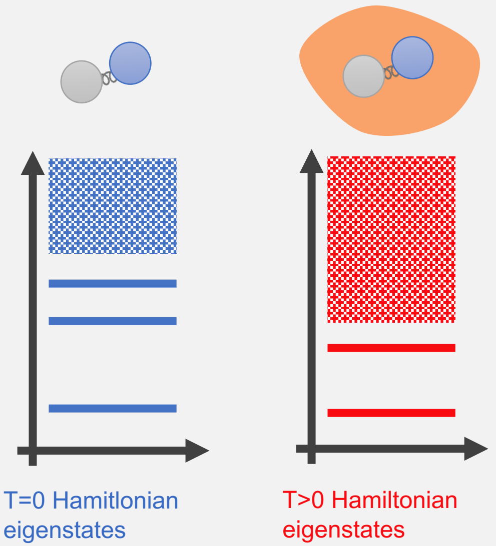
Medium effects



T=0 Hamitlonian
eigenstates

Building intuition on Quarkonium

Medium effects

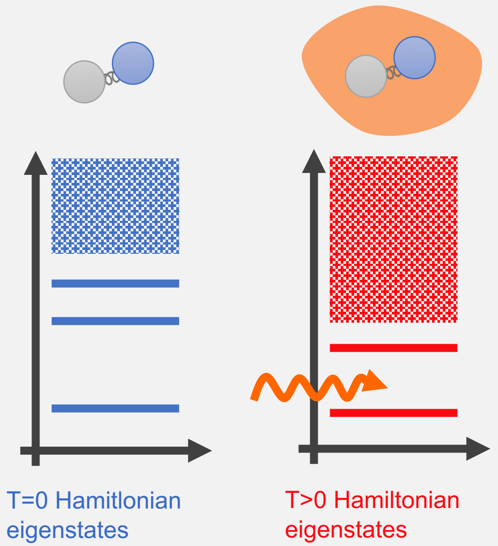


- changes QQ Hamiltonian (Matsui&Satz '86)

Building intuition on Quarkonium

University of Stavanger

Medium effects

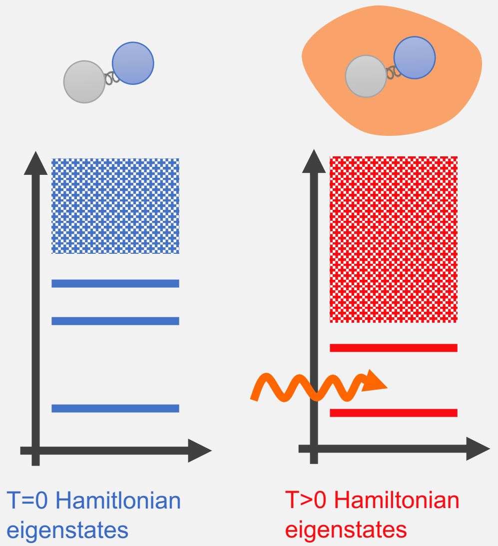


- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

Building intuition on Quarkonium

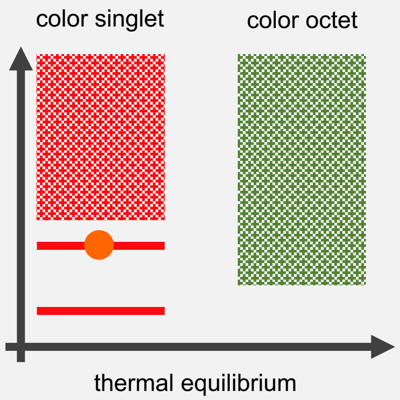
University of Stavanger

Medium effects



- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

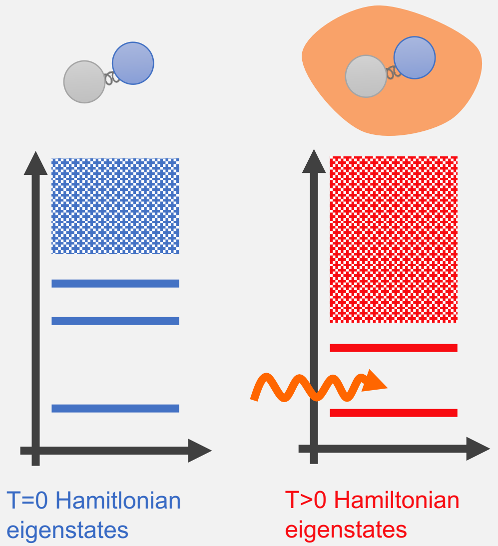
Quarkonium in-medium physics



Building intuition on Quarkonium

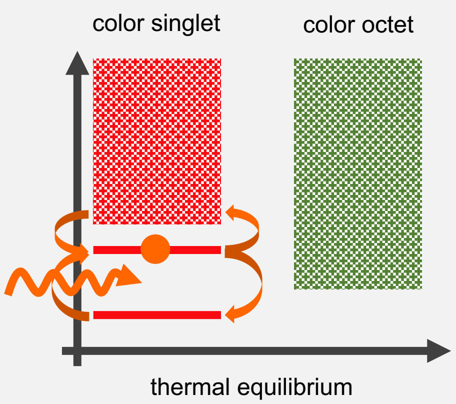
University of Stavanger

Medium effects



- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

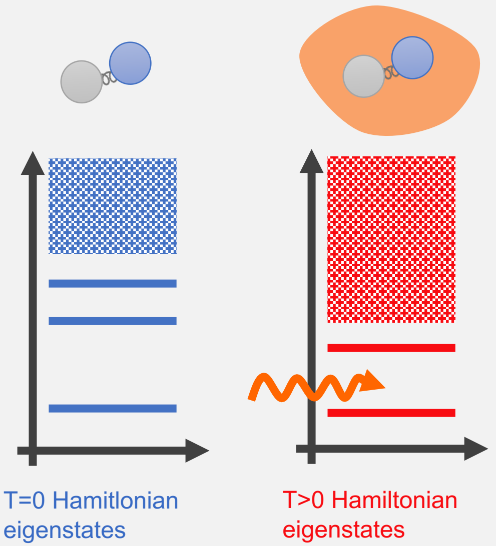
Quarkonium in-medium physics



Building intuition on Quarkonium

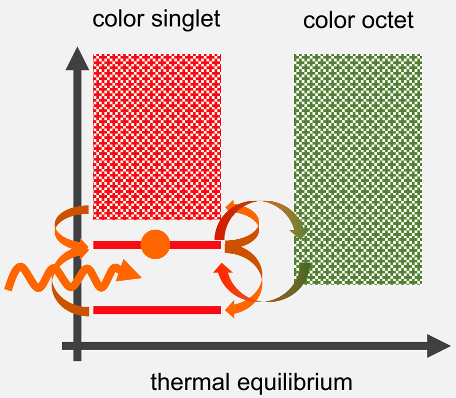
University of Stavanger

Medium effects



- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

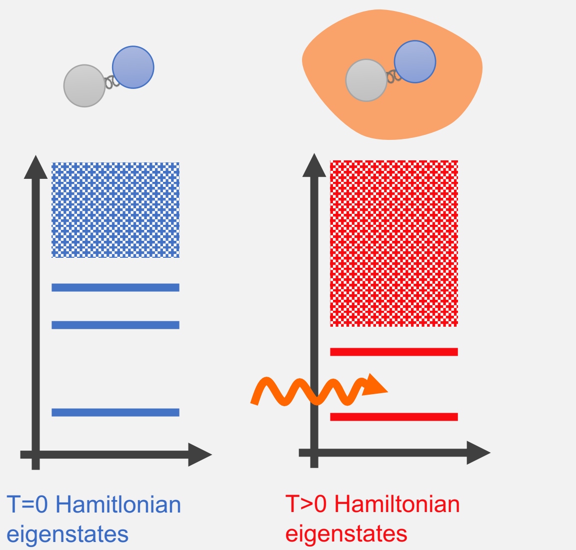
Quarkonium in-medium physics



Building intuition on Quarkonium

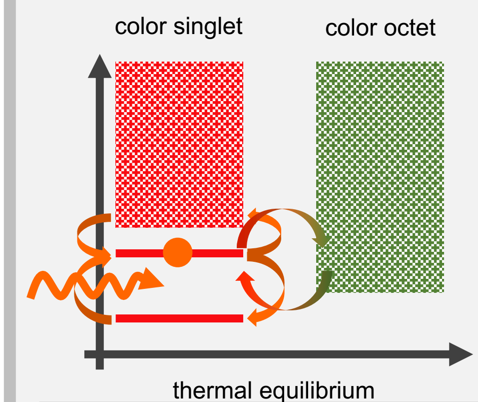
University of Stavanger

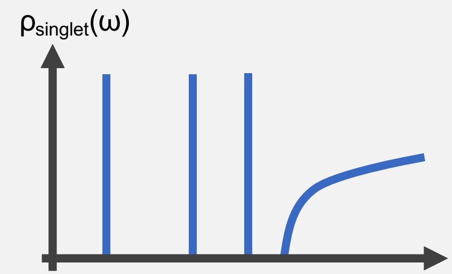
Medium effects



- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption & stimulated emission**

Quarkonium in-medium physics



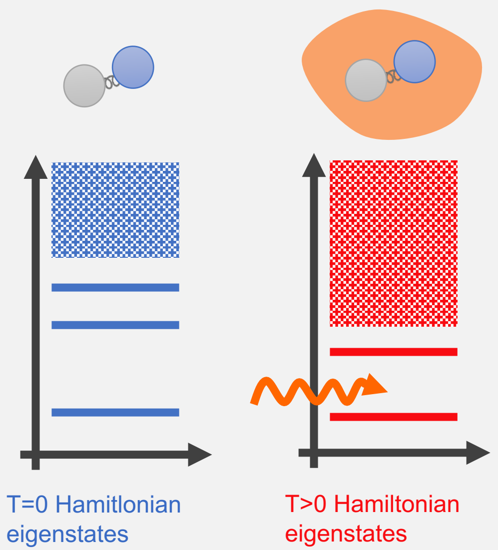


- lattice QCD **spectral function** (fully non-perturbative)

Building intuition on Quarkonium

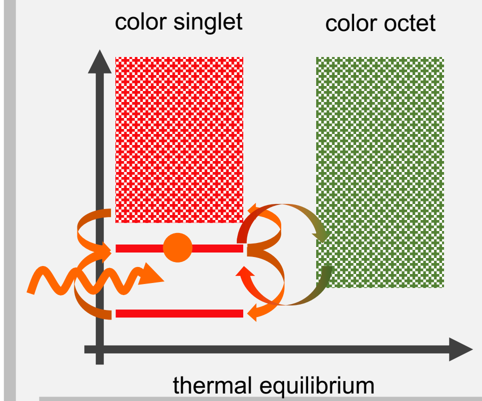
University of Stavanger

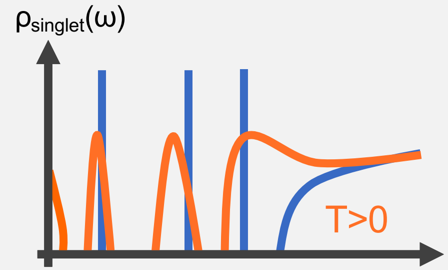
Medium effects



- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption & stimulated emission**

Quarkonium in-medium physics



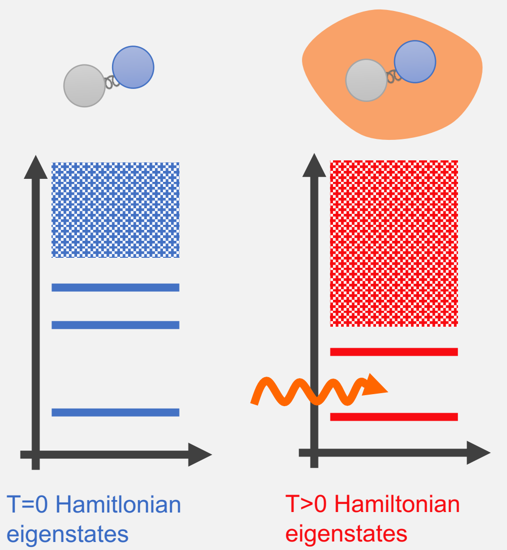


- lattice QCD **spectral function** (fully non-perturbative)

Building intuition on Quarkonium

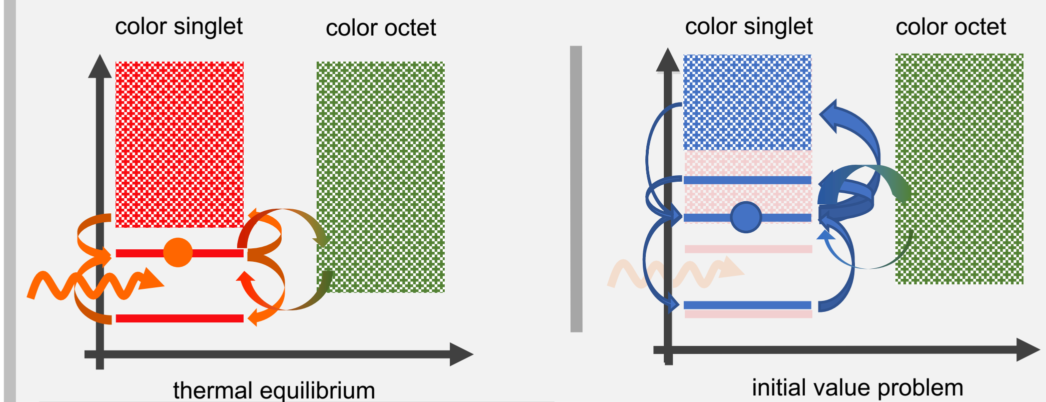
University of Stavanger

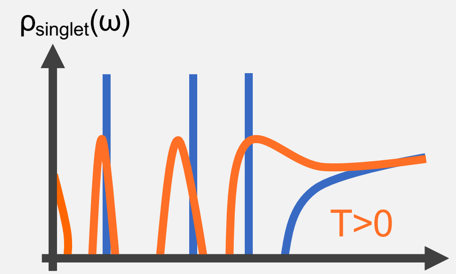
Medium effects



- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

Quarkonium in-medium physics



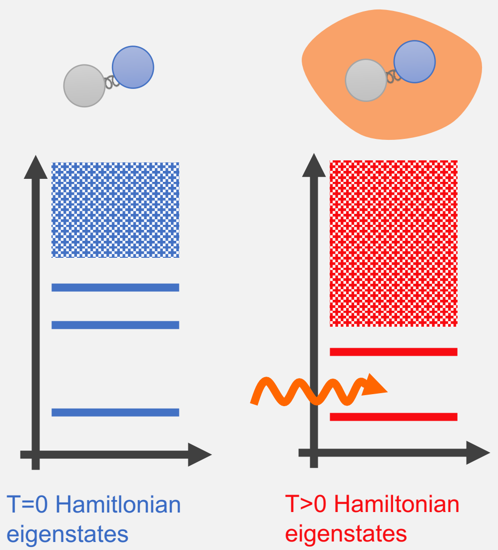


- lattice QCD **spectral function** (fully non-perturbative)

Building intuition on Quarkonium

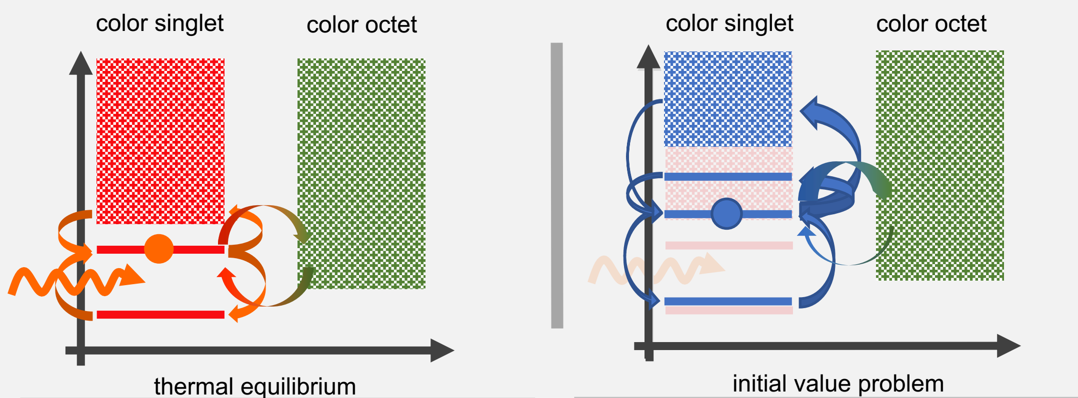
University of Stavanger

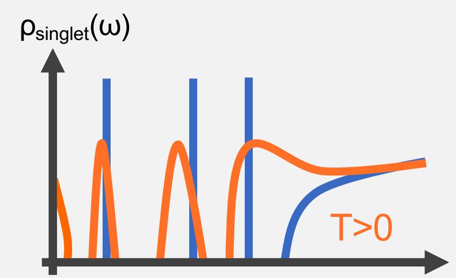
Medium effects



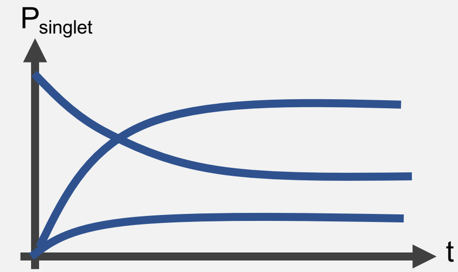
- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

Quarkonium in-medium physics





- lattice QCD **spectral function** (fully non-perturbative)

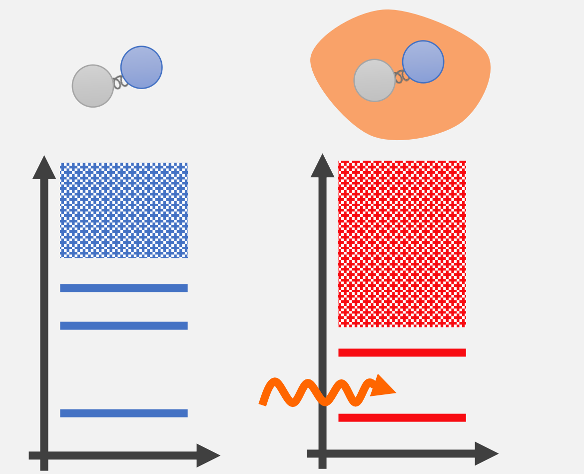


- **density matrix** master equation (non-perturbative in progress)

Building intuition on Quarkonium

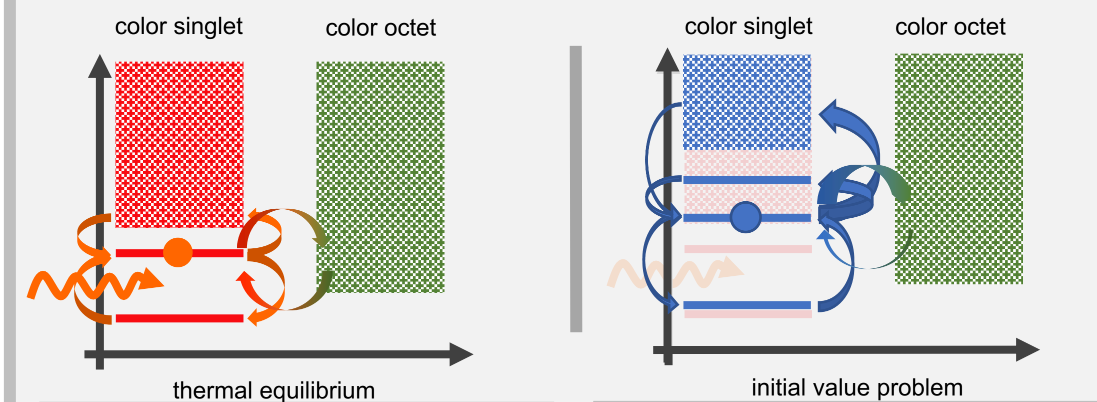
University of Stavanger

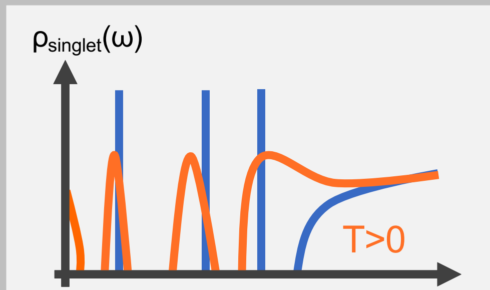
Medium effects



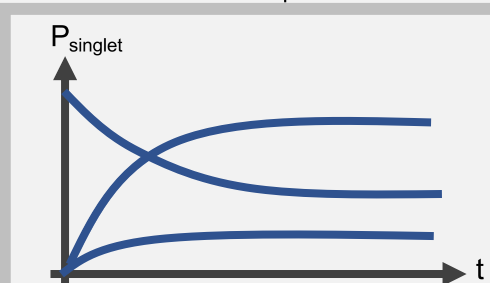
- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption & stimulated emission**

Quarkonium in-medium physics





- lattice QCD **spectral function** (fully non-perturbative)



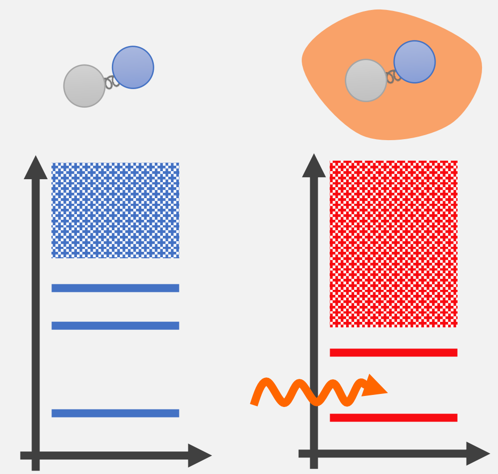
- **density matrix** master equation (non-perturbative in progress)

- Challenge I: Derive a master equation from QCD and solve it numerically

Building intuition on Quarkonium

University of Stavanger

Medium effects

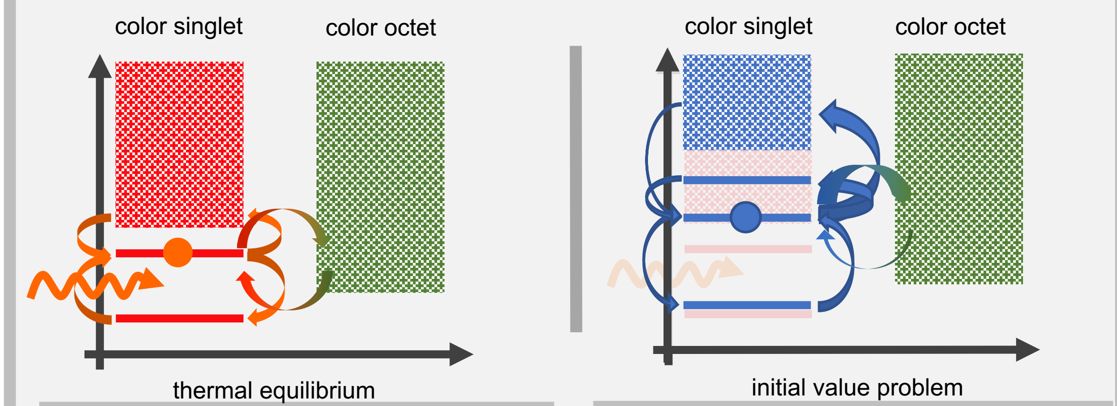


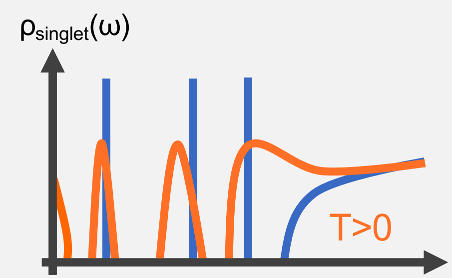
T=0 Hamiltonian eigenstates

T>0 Hamiltonian eigenstates

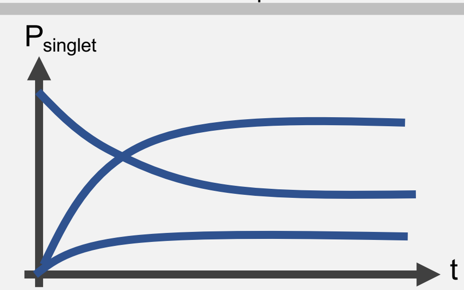
- changes QQ Hamiltonian (Matsui&Satz '86)
- acts as a radiation field: inducing **absorption** & **stimulated emission**

Quarkonium in-medium physics





- lattice QCD **spectral function** (fully non-perturbative)



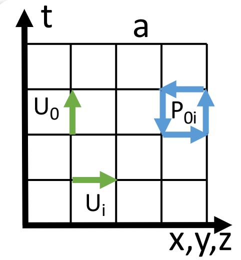
- **density matrix** master equation (non-perturbative in progress)

- Challenge I: Derive a master equation from QCD and solve it numerically
- Challenge II: Establish whether an intuitive potential picture is applicable

Classical Lattice Gauge Theory

- The Minkowski time Wilson action for SU(3)

$$S[U] = \frac{2}{g^2}\sum_x \Big\{ -\frac{a}{a_t}\sum_i \mathrm{ReTr}\Big[P_{0i} - \mathbb{1}\Big] + \frac{a_t}{a}\sum_{i<j}\mathrm{ReTr}\Big[P_{ij} - \mathbb{1}\Big]\Big\}$$



$U_\nu = \exp[igaA^a_\nu T^a]$ $\quad P_{ij} = U_i(x)U_j(x+i)U^\dagger_i(x+j)U^\dagger_j d(x)$

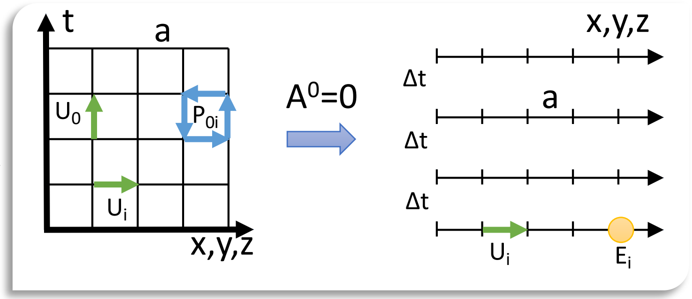
Classical Lattice Gauge Theory

University of Stavanger

- The Minkowski time Wilson action for SU(3)

$$S[U] = \frac{2}{g^2}\sum_x \left\{ -\frac{a}{a_t}\sum_i \mathrm{ReTr}\left[P_{0i} - \mathbb{1}\right] + \frac{a_t}{a}\sum_{i<j}\mathrm{ReTr}\left[P_{ij} - \mathbb{1}\right]\right\}$$

$U_\nu = \exp[igaA^a_\nu T^a]$ $\quad P_{ij} = U_i(x)U_j(x+i)U^\dagger_i(x+j)U^\dagger_j d(x)$



- In practice to solve e.o.m. turn to Hamiltonian picture

Grigriev, Rubakov Nucl. Phys. B299 1988
Ambjorn et. al. Nucl. Phys. B353 1991

  - Choose temporal gauge $U_0=1$ to decouple individual time slices
  - Degrees of freedom: SU(3) valued spatial link variables U_i and su(3) valued derivatives E_i

$$H_{\rm cl} = \sum_{\mathbf{x}}\left\{\sum_{i=1}^{3}\mathrm{Tr}\left[\mathcal{E}_i^2(\mathbf{x})\right] + \frac{1}{2N_c}\sum_{i,j=1}^{3}\mathrm{Tr}\left[\mathbb{1} - P_{ij}(\mathbf{x})\right]\right\}$$

Hamiltonian

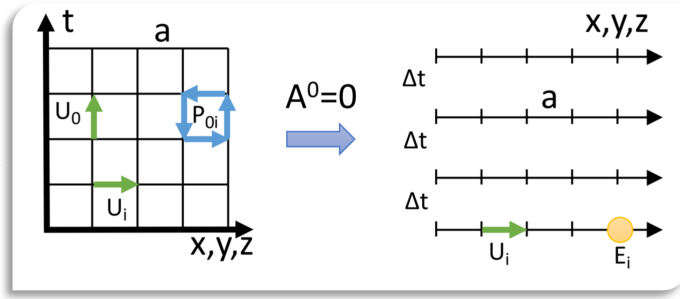
$$G(\mathbf{x},t) \equiv \sum_{\mathbf{x}}\left[E_i(\mathbf{x},t) - U_{-i}(\mathbf{x},t)E_i(\mathbf{x}-i,t)U^\dagger_{-i}(\mathbf{x},t)\right] = 0$$

Gauss constraint

Classical Lattice Gauge Theory

University of Stavanger

- The Minkowski time Wilson action for SU(3)



$$S[U]=\frac{2}{g^2}\sum_x\left\{-\frac{a}{a_t}\sum_i \mathrm{ReTr}\left[P_{0i}-\mathbb{1}\right]+\frac{a_t}{a}\sum_{i<j}\mathrm{ReTr}\left[P_{ij}-\mathbb{1}\right]\right\}$$

$U_\nu=\exp[igaA^a_\nu T^a]$ $\quad P_{ij}=U_i(x)U_j(x+i)U^\dagger_i(x+j)U^\dagger_j d(x)$

- In practice to solve e.o.m. turn to Hamiltonian picture

Grigriev, Rubakov Nucl. Phys. B299 1988
Ambjorn et. al. Nucl. Phys. B353 1991

  - Choose temporal gauge U_0=1 to decouple individual time slices
  - Degrees of freedom: SU(3) valued spatial link variables U_i and su(3) valued derivatives E_i

$$H_{\rm cl}=\sum_{\mathbf{x}}\left\{\sum_{i=1}^{3}\mathrm{Tr}\left[\mathcal{E}_i^2(\mathbf{x})\right]+\frac{1}{2N_c}\sum_{i,j=1}^{3}\mathrm{Tr}\left[\mathbb{1}-P_{ij}(\mathbf{x})\right]\right\}$$

Hamiltonian

$$G(\mathbf{x},t)\equiv\sum_{\mathbf{x}}\left[E_i(\mathbf{x},t)-U_{-i}(\mathbf{x},t)E_i(\mathbf{x}-i,t)U^\dagger_{-i}(\mathbf{x},t)\right]=0$$

Gauss constraint

  - Hamiltonian equations of motion in coordinate space:

$$a\,\partial_t U_i(\mathbf{x},t) = i\,(2C_A)^{\frac{1}{2}}\,E_i(\mathbf{x},t)U_i(\mathbf{x},t)\,,\qquad a\,\partial_t E_i^b(\mathbf{x},t) = -\left(\frac{2}{C_A}\right)^{\frac{1}{2}}\mathrm{Im\,Tr}\left[T^b U_i(\mathbf{x},t)\sum_{|j|\neq i}S^\dagger_{ij}(\mathbf{x},t)\right]$$