NOETHER CURRENTS

OF LOCALLY EQUIVALENT SYMMETRIES

AGENDA

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- LOCALLY EQUIVALENT SYMMETRIES
- EXAMPLES
- CURRENTS FROM GAUGED ACTIONS
- ENERGY-MOMENTUM TENSOR
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NOETHER CURRENTS

GLOBAL SYMMETRY : $\delta S = 0$ under some global transfo

NOETHER CURRENT : derivation à la Gell-Mann & Lévy (1960)

do a local transfo of the same action

$$\boxed{\delta S = \int dx \; J^\mu(x) \, \partial_\mu \epsilon(x)}$$

Noether current

symmetry parameter

TRIVIAL EXAMPLE

$$S = \int dx \, \frac{1}{2} (\partial_\mu \phi)^2$$

Global symmetry : $\phi \rightarrow \phi + \epsilon$

Local transformation : $\delta \phi(x) = \epsilon(x)$

$$\delta S = \int dx \, \partial^\mu \phi \, \partial_\mu \epsilon$$

$$J^\mu = \partial^\mu \phi$$

conserved on-shell :

$$\partial_\mu J^\mu = \Box \phi = 0$$

AMBIGUITIES

J^μ gives the same conservation law as $J^\mu + \tilde{J}_1^\mu + \tilde{J}_2^\mu$

vanishes on-shell

conserved off-shell

<u>Example</u>:

$$S = \int dx \, \frac{1}{2} (\partial_\mu \phi)^2$$

$$J^\mu = \partial^\mu \phi$$

$$\tilde{J}_1^\mu = \partial^\mu \phi \, \Box \phi$$

<u>Example</u>: real scalar in 2d with any S & J^μ

$$\tilde{J}_2^\mu = \epsilon^{\mu\nu} \partial_\nu \phi$$

AMBIGUITIES CONT'D

Promotion of global symmetry with ϵ to a local transformation with $\epsilon(x)$

is not unique!

Example: $S = \int dx \, \frac{1}{2} (\partial_\mu \phi)^2$

$$\delta\phi(x) = \epsilon(x) + c \, \Box \epsilon(x)$$

reduces to $\delta\phi = \epsilon$ for constant ϵ!

$$\delta S = \int dx \, \partial^\mu \phi \left(\partial_\mu \epsilon + c \, \Box \partial_\mu \epsilon \right)$$

$$J^\mu = \partial^\mu \phi + c \, \partial^\mu \Box \phi$$

gives a J_1^μ ambiguity of the current!

AMBIGUITIES CONT'D

Global symmetry : $\delta\phi_A(x) = \epsilon F_A(x)$

Localized transform : $\delta\phi_A(x) = \epsilon(x) F_A(x) + \sum_{n=1}^{\infty} \sigma_A^{\mu_1 \dots \mu_n}(x)\, \partial_{\mu_1} \cdots \partial_{\mu_n} \epsilon(x)$

gives generally

J_1^μ type ambiguity

$$J^\mu = J^\mu \Big|_{\sigma=0} + \sum_{n=1}^{\infty} (-1)^{n+1} \partial_{\mu_2} \cdots \partial_{\mu_n} \left(\sigma_A^{\mu \mu_2 \dots \mu_n} \frac{\delta S}{\delta \phi_A} \right)$$

vanishes on-shell

LOCALLY EQUIVALENT SYMMETRIES

Can two localized symmetries be made identical by relating their local coefficients?

$$\epsilon_2^a(x) = f_\alpha^a(x)\,\epsilon_1^\alpha(x)$$

$$\underbrace{\int dx\, J^\mu_{1\alpha}\,\partial_\mu \epsilon_1^\alpha}_{} \equiv \delta_1 S = \delta_2 S \equiv \int dx\, J^\mu_{2a}\,\partial_\mu \epsilon_2^a = \underbrace{\int dx\, J^\mu_{2a}\left(f^a_\alpha \partial_\mu \epsilon_1^\alpha + \epsilon_1^\alpha \partial_\mu f^a_\alpha\right)}_{}$$

These should be equal

LOCALLY EQUIVALENT SYMMETRIES

Can two localized symmetries be made identical by relating their local coefficients?

$$\epsilon_2^a(x) = f^a_\alpha(x)\,\epsilon_1^\alpha(x)$$

$$\int dx\, J^\mu_{1\alpha}\,\partial_\mu \epsilon_1^\alpha \equiv \delta_1 S = \delta_2 S \equiv \int dx\, J^\mu_{2a}\,\partial_\mu \epsilon_2^a = \int dx\, J^\mu_{2a}\left(f^a_\alpha \partial_\mu \epsilon_1^\alpha + \epsilon_1^\alpha \partial_\mu f^a_\alpha\right)$$

Consistency constraint : $J^\mu_{2a}\,\partial_\mu f^a_\alpha = \partial_\mu N^\mu_\alpha$

Current relation : $J^\mu_{1\alpha} = f^a_\alpha J^\mu_{2a} - N^\mu_\alpha$

Conservation law relation : $\partial_\mu J^\mu_{1\alpha} = f^a_\alpha\, \partial_\mu J^\mu_{2a}$

EXAMPLE 1

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2$$

locally equivalent

$$\delta\phi(x) = \epsilon_2 + \epsilon_1^\alpha x_\alpha$$

through $\epsilon_2(x) = x_\alpha \epsilon_1^\alpha(x)$

$$J_2^\mu = \partial^\mu \phi$$

Consistency constraint : $J_2^\mu \partial_\mu x_\alpha = \partial_\alpha \phi \longrightarrow N_\alpha^\mu = \delta_\alpha^\mu \phi$

Current relation : $J_{1\alpha}^\mu = x_\alpha \partial^\mu \phi - \delta_\alpha^\mu \phi$

EXAMPLE 2

Poincaré symmetry of scalar field theories

$$\phi'(x') = \phi(x)$$

$$\delta x^\mu = \epsilon_2^\mu + \frac{1}{2}\epsilon_1^{\alpha\beta}\left(x_\alpha \delta^\mu_\beta - x_\beta \delta^\mu_\alpha\right)$$

locally equivalent through

$$\epsilon_2^\mu(x) = \frac{1}{2}\epsilon_1^{\alpha\beta}(x)\left(x_\alpha \delta^\mu_\beta - x_\beta \delta^\mu_\alpha\right)$$

$$= \frac{1}{2} f^\mu_{\alpha\beta}(x)\,\epsilon_1^{\alpha\beta}(x)$$

$J_2^{\mu\nu} = T^{\mu\nu}$ ... EM tensor

Consistency constraint : $J_2{}^\mu{}_\nu\, \partial_\mu f^\nu_{\alpha\beta} = T_{\alpha\beta} - T_{\beta\alpha} = 0$ ✓

Current relation : $J^\mu_{1\,\alpha\beta} = x_\alpha T^\mu{}_\beta - x_\beta T^\mu{}_\alpha$

The relation $\vec{L} = \vec{x} \times \vec{p}$ follows directly from spacetime symmetry!

EXAMPLE 3

Symmetries of a Schrödinger scalar :

- translation : $\psi'(\vec{x}+\vec{a},t) = \psi(\vec{x},t)$
- internal U(1) : $\psi'(\vec{x},t) = e^{i\theta}\psi(\vec{x},t)$
- Galilei boost : $\psi'(\vec{x}+\vec{v}t,t) = e^{im(\vec{v}\cdot\vec{x}+\frac{1}{2}\vec{v}^2 t)}\psi(\vec{x},t)$

Galilei boost is locally equivalent to a combination of U(1) and translation!

Consistency constraint : $T^{0i} - mJ^i = \partial_\mu N^{\mu i}$

Current relation : $B^{\mu i} = tT^{\mu i} - mx^i J^\mu - N^{\mu i}$

boost current U(1) current

CURRENTS FROM GAUGED ACTIONS

Globally invariant $S[\phi]$ $\Big\}$ $\longrightarrow$ Locally invariant $\tilde{S}[\phi, A]$ such that $\tilde{S}[\phi, 0] = S[\phi]$

$$0 = \delta\tilde{S} = \delta_\phi \tilde{S} + \delta_A \tilde{S}$$

$\delta_\phi \tilde{S} \xrightarrow{A \to 0} \int dx\, J^\mu \partial_\mu \epsilon$

$\delta_A \tilde{S} \longrightarrow$ we can extract information about J^μ from here!

CURRENTS FROM GAUGED ACTIONS

Local transformation of A_μ : assume $\delta A_\mu(x) = \epsilon(x) F_\mu(x) + \sigma(x)\, \partial_\mu \epsilon(x)$

$$\delta_\phi \tilde{S} \xrightarrow{A\to 0} \int dx\, J^\mu(x)\, \partial_\mu \epsilon(x)$$

$$0 = \delta \tilde{S} = \delta_\phi \tilde{S} + \delta_A \tilde{S} \longrightarrow \int dx \frac{\delta \tilde{S}}{\delta A_\mu(x)} \left[\epsilon(x) F_\mu(x) + \sigma(x)\, \partial_\mu \epsilon(x)\right]$$

Consistency constraint :

$$\boxed{\left.\frac{\delta \tilde{S}}{\delta A_\mu(x)} F_\mu(x)\right|_{A=0} = \partial_\mu R^\mu(x)}$$

Noether current :

$$\boxed{J^\mu(x) = -\sigma(x) \left.\frac{\delta \tilde{S}}{\delta A_\mu(x)}\right|_{A=0} + R^\mu(x)}$$

EXAMPLE

$$S[\phi] = \int dx\, \frac{1}{2}(\partial_\mu \phi)^2 \longrightarrow \tilde{S}[\phi, A] = \int dx\, \frac{1}{2}(\partial_\mu \phi - A_\mu)^2$$

1) $\delta\phi(x) = \epsilon(x)$

$\delta A_\mu(x) = \partial_\mu \epsilon(x)$

Consistency : trivially satisfied

Current : $J^\mu = -\left.\dfrac{\delta \tilde{S}}{\delta A_\mu}\right|_{A=0} = \partial^\mu \phi$

2) $\delta\phi(x) = x_\alpha \epsilon^\alpha(x)$

$\delta A_\mu(x) = \epsilon_\mu(x) + x_\alpha \partial_\mu \epsilon^\alpha(x)$

$F^\alpha_\mu(x) = \delta^\alpha_\mu$

$\sigma^\alpha(x) = x^\alpha$

Consistency : $\left.\dfrac{\delta \tilde{S}}{\delta A_\mu} F^\alpha_\mu \right|_{A=0} = -\partial^\alpha \phi$

$R^\mu_\alpha(x) = -\delta^\mu_\alpha \phi$

Current : $J^\mu_\alpha = x_\alpha \partial^\mu \phi - \delta^\mu_\alpha \phi$

ENERGY – MOMENTUM TENSOR

CANONICAL : apply Noether's theorem with $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$

$\phi'_A(x') = \phi_A(x)$

$$T^{\mu}{}_{\alpha} = \delta^{\mu}_{\alpha}\mathcal{L} + \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k}\left[\partial_{\mu_2}\cdots\partial_{\mu_k}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\partial_{\mu_2}\cdots\partial_{\mu_n}\phi_A)}\right](\partial_{\mu_{k+1}}\cdots\partial_{\mu_n}\partial_{\alpha}\phi_A)$$

METRIC : couple the theory to background spacetime geometry $g_{\mu\nu}(x)$

$$\theta^{\mu\nu} = 2\left.\frac{\delta\tilde{S}}{\delta g_{\mu\nu}}\right|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

EXAMPLE – PROCA FIELD

$$\mathcal{L} = -\frac{1}{4} \underbrace{F_{\mu\nu}}_{\partial_\mu B_\nu - \partial_\nu B_\mu} F^{\mu\nu} - \frac{1}{2} m^2 B_\mu B^\mu$$

CANONICAL

$$T^{\mu\nu} = \eta^{\mu\nu} \mathcal{L} + F^{\mu\alpha} \partial^\nu B_\alpha$$

METRIC

$$\Theta^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha} F^{\nu}{}_{\alpha} + m^2 B^\mu B^\nu$$

$$= T^{\mu\nu} + \underbrace{B^\nu \left(\partial_\alpha F^{\mu\alpha} + m^2 B^\mu\right)}_{\text{vanishes on-shell}} - \underbrace{\partial_\alpha \left(F^{\mu\alpha} B^\nu\right)}_{\text{conserved off-shell}}$$

EXAMPLE – PROCA FIELD

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 B_\mu B^\mu$$

$$\partial_\mu B_\nu - \partial_\nu B_\mu$$

CANONICAL

$$T^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}\partial^\nu B_\alpha$$

WHAT IS GOING ON HERE ?!

METRIC

$$\Theta^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}F^{\nu}{}_{\alpha} + m^2 B^\mu B^\nu$$

$$= T^{\mu\nu} + \underbrace{B^\nu\left(\partial_\alpha F^{\mu\alpha} + m^2 B^\mu\right)}_{\text{vanishes on-shell}} - \underbrace{\partial_\alpha\left(F^{\mu\alpha}B^\nu\right)}_{\text{conserved off-shell}}$$

EXAMPLE – PROCA FIELD

CANONICAL

$$T^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}\partial^{\nu}B_{\alpha}$$

Based on

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$$

$$B'_{\mu}(x') = B_{\mu}(x)$$

!

METRIC

$$\Theta^{\mu\nu} = \eta^{\mu\nu}\mathcal{L} + F^{\mu\alpha}F^{\nu}{}_{\alpha} + m^2 B^{\mu}B^{\nu}$$

Based on

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$$

$$B'_{\mu}(x') = B_{\mu}(x) - B_{\alpha}(x)\,\partial_{\mu}\epsilon^{\alpha}(x)$$

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - g_{\alpha\nu}(x)\,\partial_{\mu}\epsilon^{\alpha}(x) - g_{\mu\alpha}(x)\,\partial_{\nu}\epsilon^{\alpha}(x)$$

METRIC EMT FROM NOETHER'S THEOREM

Take $B'_\mu(x') = B_\mu(x) \underbrace{- B_\alpha(x)\,\partial_\mu \epsilon^\alpha(x)}_{\sigma^\nu_{\mu\alpha}(x)\,\partial_\nu \epsilon^\alpha(x)}$ seriously!

New EM tensor:

$$\tilde{T}^{\mu\nu} = T^{\mu\nu}\Big|_{\sigma=0} + \eta^{\nu\alpha}\sigma^{\mu}_{\lambda\alpha}\frac{\delta S}{\delta B_\lambda}$$

$$= T^{\mu\nu} + B^\nu\left(\partial_\alpha F^{\mu\alpha} + m^2 B^\mu\right) = \Theta^{\mu\nu} + \underbrace{\partial_\alpha\left(F^{\mu\alpha}B^\nu\right)}_{\text{Conserved off-shell}}$$

We have removed the on-shell ambiguity!

CANONICAL EMT FROM GAUGED ACTION

Gauge the action while preserving $B'_\mu(x') = B_\mu(x)$!

Use vielbein instead of metric : $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$.

$$S[B] \longrightarrow \tilde{S}[B,e]$$

$$T^\mu{}_\alpha = e^a_\alpha \left. \frac{\delta \tilde{S}}{\delta e^a_\mu} \right|_{e^a_\mu = \delta^a_\mu}$$

Trade B_μ for a set of scalars : $B_a = e^\mu_a B_\mu$

$$\tilde{S}[B,e] = \int dx \, \|e\| \left(-\frac{1}{4} \eta^{ac} \eta^{bd} F_{ab} F_{cd} - \frac{1}{2} m^2 \eta^{ab} B_a B_b \right)$$

$F_{ab} = e^\mu_a \partial_\mu B_b - e^\mu_b \partial_\mu B_a$

These together correctly reproduce $T^{\mu\nu}$!

MORALS 1

Relations between currents mirror relations between symmetries :

$$\epsilon_2^a(x) = f_\alpha^a(x)\,\epsilon_1^\alpha(x) \longrightarrow J_{1\alpha}^\mu = f_\alpha^a J_{2a}^\mu - N_\alpha^\mu$$

What is this good for ?

- Better understanding of consequences of symmetries.
- Absence of Goldstone bosons for "redundant" symmetries.
- Enhanced soft limit of scattering amplitudes of massless scalars.

MORALS 2

Canonical & metric EM tensors can be reconciled by taking translation invariance seriously!

- No need for ad hoc "improvement" of the EM tensor.
- No need to invoke gauge invariance.

MORALS 2'

The difference between canonical & metric EM tensors may be physically significant!

chemical potential

Field theory at finite density : $\partial_0\phi \longrightarrow (\partial_0 - i\mu)\phi$

How about making this covariant : $\partial_\mu\phi \longrightarrow D_\mu\phi = (\partial_\mu - iA_\mu)\phi$

$$S[\phi] \longrightarrow \tilde{S}[\phi, A, g]$$

$$\theta^{\mu\nu} = 2\frac{\delta\tilde{S}}{\delta g_{\mu\nu}}\Big|_{g=\eta} \longrightarrow$$

grandcanonical Hamiltonian θ^{00}?

MORALS 2'

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$$\theta^{\mu\nu} = 2\frac{\delta \tilde{S}}{\delta g_{\mu\nu}}\bigg|_{g=\eta} \longrightarrow$$

grandcanonical Hamiltonian θ^{00}?

DO NOT DO THIS!

THANK YOU!