# QFT and Invariants of Exotic Geometries

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# DISCLAIMER



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- What is it good for, if not testable or even a description of reality?
- If not physics, then mathematics?
- How can methods of physics with no rigorous mathematical basis (e.g. path integral) be used to derive mathematically sound results?

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Tentative responses:

- The physics often leads one towards deep conjectures about geometry (e.g. Mirror Symmetry).
- Though the methods are mathematically iffy, the end results might still make sense (e.g. well-defined invariants).
- Personal hope: To connect the mathematics with phenomenological model building in particle physics and string phenomenology.

# Introduction

# Kaluza-Klein and String Compactifications

String Theory is a higher dimensional UV-completion for the unification of the fundamental forces (including gravity). Playground for KK thories.

String Phenomenology: We postulate a space-time of the form

$$M_D = M_4 \times X_n \ , \quad \mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + g_{ab}\mathrm{d}x^a \mathrm{d}x^b$$

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Idea:

- Deformations of $X_n$ geometry $\leftrightarrow$ Fields in 4d geometry.
- Look for 4d theories containing the Standard Model spectrum.
- Need mechanism to lift additional massless fields out of theory (moduli stabilisation). **This is the difficult bit!**
- In particular, very hard to cook up models without runaway directions where $X_n$ decompactifies.
- Geometric invariants can be useful in this regard.

# Invariants and Threshold Corrections

# Threshold Corrections



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Threshold corrections:

- Such internal quantum effect are colloquially called threshold corrections.
- Often required to get the right RG-flow of effective field theory couplings, etc.
- Might help with moduli stabilisation: Generate non-perturbative couplings in effective physics.
- Drawback: Often hard to compute.

# Topological QFT and invariants



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The gist:

- The internal space $X_n$ is equipped with some geometric structure $\Psi$ (gauge-field, metric, complex structure, etc).
- The geometric structure $\Psi$ is governed by some topological action $S(\Psi)$ deriving from the (stringy) higher-dimensional supergravity.
- Moduli space: Space of solutions of $\Psi$ to equations of motion $\delta S(\Psi) = 0$.

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Quantum corrections: We quatize $S(\Psi)$

$$Z(X_n) = \int \mathcal{D}\Psi \, e^{iS(\Psi)}$$

Note: This is a **topological invariant** associated to the geometry $X_n$ (integrate over geometric structures). Threshold corrections are associated to such topological invariants of internal geometries.

# Example: Chern-Simons Theory



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CS theory can be defined for a (curved) three-dimensional manifold $M_3$, equipped with a gauge theory and associated gauge connection $A_a$:

$$S_{CS}(A) = \int_{M_3} \text{tr}\left(A_a \partial_b A_c + \tfrac{2}{3} A_a A_b A_c\right) \epsilon^{abc} \ .$$

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Equations of motion (suppressed gauge indecies):

$$F_{ab} = \partial_{[a} A_{b]} + A_{[a} A_{b]} = 0 .$$

These correspond to flat gauge bundles.

The moduli space of flat gauge bundles on a compact space is finite-dimensional. Gives rise to a finite number of fields in the effective theory.

## Example: Chern-Simons Theory



We define the partition function for a gauge group $G$ (e.g. $G = SU(N)$)

$$Z(M_3, G) = \int \mathcal{D}A\, e^{iS(A)} \,.$$

No metric dependence $\Rightarrow Z(M_3, G)$ should be an **invariant** [Witten '89, Reshetikhin-Turaev '91], depending only on the topological date of $M_3$ and the gauge group $G$.

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One can use $Z(M_3, G)$ to extract other interesting invariants of $M_3$. E.g. the one-loop partition function [Witten '89], and:

- A perturbative computation of $Z(M_3, G)$ leads to the definition of **universal perturbative invariants**, independent of the gauge group [Axelrod-Singer '91].
- CS theory is related to the topological open string [Witten '92] $\Rightarrow$ relations to Gromov-Witten invariants by open-closed string duality [Vafa '01, ..]..
- Expectation values of Wilson loops: **Knot invariants** proportional to the **Jones Polynomial** of the knot [Witten '89, ..].

The study of quantum CS theory is a fruitful endeavour to this date.

# Other examples



There are many other generalisation and (quasi) topological theories which deserve mentioning:

- Holomorphic Chern-Simons theory (on six-dimensional Calabi-Yau manifolds): Give rise to Donaldson-Thomas invariants.
- Different types of volume (Hitchin) functionals in ordinary, generalised and exceptional geometry.
- Kodaira-Spencer gravity and Wittens topological strings.
- Generalisations combining gauge and gravitational degrees of freedom (my own work in heterotic string theory).

These all appear in different parts of string geometry, sometimes with equivalences (string dualities) between them.

Explicit computation (resulting in actual numbers) for invariants can be very tricky.

Exact methods in QFT such as localisation can be useful. Usually requires extended supersymmetry implying a more rigid geometric structure.

**Localisation:** Reduces formal expression (path integrals) to integrals over finite dimensional moduli space of classical solutions which can in principle be performed (and sometimes is).

# Computing Invariants

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**Localisation:** Reduces formal expression (path integrals) to integrals over finite dimensional moduli space of classical solutions which can in principle be performed (and sometimes is).

**First step:** One-loop computation. Compute the partition function of the quadratic approximation to the action. This action Is usually of the form

$$S(\Psi) = \int_{X_n} \delta\Psi^* \Delta \delta\Psi + .. ,$$

where $\Delta$ is generically some elliptic linear operator.

**Result:** Ray–Singer torsion of $\Delta$. This is often a topological invariant, but can have gravitational anomalies interesting in their own right.

# Conclusions

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Let's recall the main points of the talk:

- There is a vibrant mathematical community studying geometric invariants.
- These invariants can be defined, and often computed using methods of QFT.
- They can be relevant for phenomenology, and string model building in particular.
- Can be hard to compute explicitly (numerically??).
- Often necessary to resort to special cases with extended supersymmetry and rigid geometric structures where exact methods of QFT can be employed.

# Thank you!



Thank you for your attention!